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## Near-Wall Modeling of Compressible Turbulent Flows *8.17*

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## Summary

A near-wall two-equation model for compressible flows is proposed. The model is formulated by relaxing the assumption of dynamical field similarity between compressible and incompressible flows. A postulate is made to justify the extension of incompressible models to account for compressibility effects. This requires formulating the turbulent kinetic energy equation in a form similar to its incompressible counterpart. As a result, the compressible dissipation function has to be split into a solenoidal part, which is not sensitive to changes of compressibility indicators, and a dilatational part, which is directly affected by these changes. This procedure, therefore, isolates terms with explicit dependence on compressibility so that they can be modeled accordingly. An equation that governs the transport of the solenoidal dissipation rate with additional terms that are explicitly dependent on compressibility effects is derived similarly. A model with an explicit dependence on the turbulent Mach number is proposed for the dilatational dissipation rate. Thus formulated, all near-wall incompressible flow models could be expressed in terms of the solenoidal dissipation rate and straight-forwardly extended to compressible flows. Therefore, the incompressible equations are recovered correctly in the limit of constant density. The two-equation model and the assumption of constant turbulent Prandtl number are used to calculate compressible boundary layers on a flat plate with different wall thermal boundary conditions and free-stream Mach numbers. The calculated results, including the near-wall distributions of turbulence statistics and their limiting behavior, are in good agreement with measurements. In particular, the near-wall asymptotic properties are found to be consistent with incompressible behavior; thus suggesting that turbulent flows in the viscous sublayer are not much affected by compressibility effects.

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## 1. Introduction

Density variation in a turbulent flow can come from different sources. Some of these are: (i) isothermal mixing of gases of different density, (ii) strong temperature gradient in a homogeneous fluid, (iii) reactive flows and (iv) compressibility effects in high speed flows. Each of these sources gives rise to specific aspects that require modeling if the governing equations are to be solved. This study makes an attempt to address the last source; that is, the modeling of high speed compressible turbulent flows.

Most studies on compressible turbulent flow modeling<sup>1-9</sup> invoke the Morkovin postulate<sup>10</sup> to justify the direct extension of the incompressible models to compressible flows. The postulate was formulated based on early experiments on compressible boundary layers along adiabatic walls and compressible wakes, and essentially suggested that the dynamical field in a compressible flow behaves like an incompressible one. This postulate was used by numerous researchers to assure that compressibility effects can be accounted for directly by the variable mean density in the governing equations alone. In other words, the influences of fluctuating density on turbulence mixing are essentially assumed to be negligible. The validity and extent of Morkovin's postulate were reviewed by Bradshaw<sup>2</sup> and he noted that the postulate is appropriate for flows where density fluctuations are moderate. Therefore, the postulate is not valid for hypersonic boundary layers, where the Mach number is five or greater, and for flows with strong pressure gradient effects, such as shock-turbulent-boundary-layer interactions. The latter point was confirmed by the studies of Wilcox and Alber<sup>1</sup> and Bradshaw<sup>11</sup> and led to proposals to have the effects of pressure-dilatation correlation modeled in the governing equations<sup>12</sup>. A more recent study where density fluctuations are also considered has been given by Speziale and Sarkar<sup>13</sup>. Besides these modifications, all turbulent compressible flow modeling rely on incompressible models.

Two sources of difficulties arise when incompressible turbulence models are extended to compressible flows. One is due to compressibility itself and another is associated with the turbulence phenomena. In compressible flows, the governing equations are coupled and temperature cannot be considered as a passive scalar. As a result, all other thermodynamic variables adopt new roles. Therefore, mathematically, compressible flows cannot be considered as straightforward extension of incompressible flows. Furthermore, pressure is only a force term in incompressible flows and all disturbances propagate at infinite speed. On the other hand, pressure also supports finite velocity propagation of disturbances in compressible flows. Other complications come from the variable mean density, which contributes to increased non-linearity of the governing equations, and the fluctuating density, which causes the closure problem to become more difficult.

The second source of difficulties has to do with turbulence mixing. Here, even for incompressible flows, many problems remain to be resolved<sup>14-17</sup>, especially when the flow is unsteady and/or three-dimensional<sup>18</sup>. However, among the many problems associated with turbulence modeling, one stands out as most fundamental and urgently needs attention. This is the treatment of the near-wall flow<sup>17</sup>. Conventional approach is to invoke the wall function assumptions; thus implying that near-wall turbulence is in local equilibrium. Even for simple wall shear flows, the assumption is not quite valid because near-wall turbulence is not in local equilibrium. Consequently, a low-Reynolds-number treatment is necessary in order to obtain results that agree with measurements in the near-wall region<sup>17,19-21</sup>. The need for near-wall treatment of flows with heat and mass transfer has also been pointed out<sup>22-25</sup>. This problem is expected to be more acute in compressible flow modeling<sup>13</sup> where the non-linearity of the governing equations are further compounded by the variable mean density.

The present objective is to model near-wall compressible turbulent flows where the coupling between velocity and temperature cannot be ignored. As a first attempt, only the

modeling of the turbulent kinetic energy and its dissipation-rate equations is considered. With the assumption of gradient transport, the two-equation model could be used to effect closure of the mean flow equations. Since the transport equations for the heat fluxes and the temperature dissipation rate are not modeled and solved, a constant turbulent Prandtl number is invoked to relate the heat fluxes to the momentum fluxes. In view of this assumption, the present approach only addresses the issue of compressibility effects on turbulent mixing and not on heat transfer and its interaction with turbulence. An attempt on this latter problem will be made after the present model has been validated.

## 2. Proposed Modeling Approach

With the availability of near-wall models for temperature variance and its dissipation rate<sup>24</sup>, heat fluxes<sup>25</sup>, Reynolds-stresses<sup>26</sup> and the dissipation rate of the turbulent kinetic energy<sup>27</sup>, the time is now ripe for their extension to compressible flows. In order to consider the effects of variable mean density and its fluctuation on turbulence mixing, it is necessary to analyse the exact equations and propose appropriate models to effect closure. Two approaches are available. One is to propose totally new models for the terms in the compressible equations, while another is to attempt to extend the incompressible models to compressible flows in a credible way. Both approaches involve assumptions that could or could not be verified experimentally. Since the present knowledge of incompressible flow modeling is quite mature, as a first attempt, it is expedient to extend these models to compressible flows. This can be accomplished by recasting the compressible equations in forms similar to their incompressible counterparts so that terms with explicit dependence on compressibility effects can be isolated separately, and the incompressible limit can be recovered in a straight forward and correct manner.

Since the turbulent kinetic energy equation or k-equation is obtained by contracting the Reynolds-stress equations, this means that the recasting of the Reynolds-stress equations should be attempted first. In other words, the viscous diffusion and dissipation terms in the Reynolds-stress equations have to be similarly defined as their incompressible counterparts. This suggests splitting the viscous dissipation function into a solenoidal part, which is not sensitive to changes of compressibility indicators, and a dilatational part, which is directly affected by these changes<sup>28</sup>. When the Reynolds-stress equations are written in this form, three additional terms that depend explicitly on compressibility effects are present. The k-equation is then obtained by contracting the Reynolds-stress equations and its incompressible counterpart is recovered correctly when density becomes constant and the additional terms vanish identically. An equation that governs the transport of the solenoidal dissipation rate ( $\epsilon$ ) of the turbulent kinetic energy ( $k$ ) is derived and

modeled along the line suggested above. Again, additional terms that depend explicitly on compressibility effects appear in the equation. This equation also reduces correctly to its modeled incompressible counterpart because the additional terms vanish for constant density flows. All models proposed for the  $k$  and  $\epsilon$  equations are expressed in terms of this solenoidal dissipation rate. A model with explicit dependence on the turbulent Mach number proposed by Sarkar et al.<sup>28</sup> for the dilatational dissipation is adopted. Thus formulated, the two-equation model is valid for compressible flows and approaches its incompressible limit in a straight forward and correct manner.

The systematic approach described above, if proven successful, could be used to extend incompressible near-wall models for heat-fluxes, temperature variance and its dissipation rate to compressible flows. A set of equations governing the transport of incompressible heat fluxes has been proposed and validated against simple flows with heat transfer<sup>25</sup>, while a similar set of equations for the temperature variance and its dissipation rate<sup>24</sup> has also been validated against boundary-layer flows. This means that near-wall heat transfer models could also be extended to compressible flows using the approach proposed above. Until such time, the assumption of a constant turbulent Prandtl number for near-wall compressible flow is inevitable.

In the following, the compressible equations are first derived, then the near-wall modeling of the  $k$  and  $\epsilon$  equations are discussed. In section 6, the two-equation model is used to calculate compressible boundary layers on a flat plate assuming a constant turbulent Prandtl number. The calculations are carried out for a range of free-stream Mach number and two different wall thermal boundary conditions. Comparisons with measurements<sup>9,29-31</sup> and other calculations, such as those obtained using the  $k$ - $\omega$  model of Wilcox<sup>8</sup>, are carried out to assess the importance of density fluctuations on the calculated results and, hence, the validity and extent of Morkovin's hypothesis.



### 3. Mean Flow Equations

The compressible mean flow equations are obtained by applying Favre averaging to the instantaneous Navier-Stokes equations which for Newtonian fluids can be written as:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i} (\rho u_i) = 0 \quad , \quad (1)$$

$$\frac{\partial (\rho u_i)}{\partial t} + \frac{\partial}{\partial x_j} (\rho u_i u_j) = - \frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} \quad , \quad (2)$$

$$\frac{\partial (\rho C_p T)}{\partial t} + \frac{\partial}{\partial x_i} (\rho C_p T u_i) = \frac{\partial p}{\partial t} + u_i \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x} \left( \kappa \frac{\partial T}{\partial x_i} \right) + \tau_{ij} \frac{\partial u_i}{\partial x_j} \quad , \quad (3)$$

$$\text{where } \tau_{ij} = \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{2}{3} \mu \frac{\partial u_k}{\partial x_k} \delta_{ij} \quad , \quad (4)$$

$u_i$  is the  $i^{\text{th}}$  component of the velocity vector,  $x_i$  is the  $i^{\text{th}}$  component of the coordinates and  $p$ ,  $T$ ,  $\rho$ ,  $\mu$ ,  $\kappa$ ,  $C_p$  are pressure, temperature, density, viscosity, thermal conductivity and specific heat at constant pressure, respectively. Favre decomposition is applied to all variables except  $p$  and  $\rho$  where conventional Reynolds decomposition is assumed. In other words

$$u_i = \langle U_i \rangle + u_i'' \quad , \quad (5)$$

$$T = \langle \Theta \rangle + \theta'' \quad , \quad (6)$$

$$p = \bar{P} + p' \quad , \quad (7)$$

$$\rho = \bar{\rho} + \rho' \quad , \quad (8)$$

where  $u_i''$  and  $\theta''$  are the Favre fluctuations and  $p'$  and  $\rho'$  are the Reynolds fluctuations. If  $\langle \rangle$  is used to denote Favre-averaged quantities and the overbar the Reynolds-averaged quantities, then the mean equations for compressible flows can be obtained as follows. The above decompositions (5)-(8) are substituted into (1)-(4) and the resultant equations are averaged over time. If the turbulent flow is further assumed to be stationary and the mean momentum equation and the

Reynolds-stress and turbulent kinetic energy,  $k = \frac{1}{2}\langle u_i'' u_i'' \rangle$ , equations to be derived later are used to simplify the thermal energy equation, the turbulent mean flow equations become

$$\frac{\partial}{\partial x_i} (\bar{\rho} \langle U_i \rangle) = 0 \quad , \quad (9)$$

$$\frac{\partial}{\partial x_j} (\bar{\rho} \langle U_i \rangle \langle U_j \rangle) = - \frac{\partial \bar{P}}{\partial x_i} + \frac{\partial \langle \tau_{ij} \rangle}{\partial x_j} - \frac{\partial}{\partial x_j} (\bar{\rho} \langle u_i'' \rangle \langle u_j'' \rangle) + \underline{\frac{\partial \tau_{ij}''}{\partial x_j}} \quad , \quad (10)$$

$$\begin{aligned} \frac{\partial}{\partial x_i} (\bar{\rho} \langle U_i \rangle [\bar{C}_p \langle \Theta \rangle + \frac{1}{2} \langle U_k \rangle \langle U_k \rangle + k]) &= \frac{\partial}{\partial x_i} \left( \bar{\kappa} \frac{\partial \langle \Theta \rangle}{\partial x_i} \right) \\ &+ \frac{\partial}{\partial x_j} (\langle \tau_{ij} \rangle \langle U_i \rangle) - \frac{\partial}{\partial x_i} (\bar{\rho} \bar{C}_p \langle \theta'' u_i'' \rangle) - \frac{\partial}{\partial x_i} (\bar{\rho} \langle u_i'' u_j'' \rangle \langle U_j \rangle) \\ &+ \underline{\frac{\partial}{\partial x_i} (\tau_{ij}'' \langle U_j \rangle)} - \underline{\frac{\partial}{\partial x_i} (\bar{\rho} \langle k u_i'' \rangle)} + \underline{\frac{\partial}{\partial x_i} (u_j'' \tau_{ij}'')} + \underline{\frac{\partial}{\partial x_i} (u_j'' \langle \tau_{ij} \rangle)} + \underline{\frac{\partial}{\partial x_i} \left( \bar{\kappa} \frac{\partial \theta''}{\partial x_i} \right)} \quad . \end{aligned} \quad (11)$$

In these equations,  $\mu = \bar{\mu}$ ,  $\kappa = \bar{\kappa}$  and  $C_p = \bar{C}_p$  have been substituted and the mean and fluctuating stresses are given by

$$\begin{aligned} \langle \tau_{ij} \rangle &= \bar{\mu} \left( \frac{\partial \langle U_i \rangle}{\partial x_j} + \frac{\partial \langle U_j \rangle}{\partial x_i} \right) - \frac{2}{3} \bar{\mu} \delta_{ij} \frac{\partial \langle U_k \rangle}{\partial x_k} \quad , \\ \tau_{ij}'' &= \bar{\mu} \left( \frac{\partial u_i''}{\partial x_j} + \frac{\partial u_j''}{\partial x_i} \right) - \frac{2}{3} \bar{\mu} \delta_{ij} \frac{\partial u_k''}{\partial x_k} \quad . \end{aligned}$$

The quantity,  $\bar{C}_p \langle \Theta \rangle + \frac{1}{2} \langle U_k \rangle \langle U_k \rangle + k$ , is the mean total enthalpy  $\langle H \rangle$ . Thus written, (9)-(11) reduce to their incompressible counterparts exactly when density becomes constant.

An order-of-magnitude analysis is carried out on (9) - (11). The result shows that the underlined terms are of smaller order and, as a first approximation, could be neglected compared to the terms retained. Thus formulated, the compressible equations are identical to the incompressible equations and the additional unknowns are the turbulent momentum and heat fluxes, just as in the

incompressible case. The present approach proposes to close these equations assuming gradient transport. As a first attempt, a near-wall two-equation  $k$ - $\epsilon$  model is used to determine the turbulent viscosity and a constant turbulent Prandtl number is invoked to relate turbulent momentum and heat fluxes. Therefore, the present model cannot fully account for the effects of density fluctuation on turbulent heat transfer.

#### 4. Modeling of the Turbulent Kinetic Energy Equation

The Favre-averaged transport equation for the Reynolds stresses  $\overline{\rho \langle u_i' u_j' \rangle}$  could be similarly derived as in the incompressible case<sup>16</sup>. That is, the *i*th fluctuating velocity equation is obtained by subtracting the mean momentum equation from the instantaneous equation. Repeat the same procedure to obtain the *j*th fluctuating velocity equation. The *i*th fluctuating velocity equation is then multiplied by the *j*th fluctuation velocity and vice versa and the two equations are then added together and averaged over time. Omitting all the algebra, the final exact equation is:

$$\begin{aligned} \frac{\partial}{\partial t} \left[ \overline{\rho \langle u_i' u_j' \rangle} \right] + \frac{\partial}{\partial x_k} \left[ \overline{\rho \langle U_k \rangle \langle u_i' u_j' \rangle} \right] = & - \frac{\partial}{\partial x_k} \left[ \overline{\rho \langle u_i' u_j' u_k' \rangle} \right] + \frac{\partial}{\partial x_k} \left[ \overline{u_i' \tau_{jk}'} + \overline{u_j' \tau_{ik}'} \right] \\ & - \left[ \overline{\tau_{ik}'} \frac{\partial u_j'}{\partial x_k} + \overline{\tau_{jk}'} \frac{\partial u_i'}{\partial x_k} \right] - \left[ \overline{u_i' \frac{\partial p'}{\partial x_j}} + \overline{u_j' \frac{\partial p'}{\partial x_i}} \right] - \left[ \overline{\rho \langle u_i' u_k' \rangle} \frac{\partial \langle U_j \rangle}{\partial x_k} + \overline{\rho \langle u_j' u_k' \rangle} \frac{\partial \langle U_i \rangle}{\partial x_k} \right] \\ & - \left[ \overline{u_i' \frac{\partial \overline{P}}{\partial x_j}} + \overline{u_j' \frac{\partial \overline{P}}{\partial x_i}} \right] + \left[ \overline{u_i' \frac{\partial \langle \tau_{jk} \rangle}{\partial x_k}} + \overline{u_j' \frac{\partial \langle \tau_{ik} \rangle}{\partial x_k}} \right] . \end{aligned} \quad (12)$$

Symbolically, the above equation can be written as

$$C_{ij} = D_{ij}^T + D_{ij}^{*v} - \overline{\rho} \epsilon_{ij}^* + \Phi_{ij} + P_{ij} + G_{ij} + T_{ij} . \quad (13)$$

With the exception of  $G_{ij}$  and  $T_{ij}$ , (13) is similar to its incompressible counterpart<sup>26</sup>. For an incompressible flow,  $\overline{u_i'} \equiv 0$ , and  $G_{ij} = T_{ij} = 0$ . Even under this condition, (13) fails to reduce properly to the incompressible equation given in Ref. 26. The reason lies in the grouping of the terms  $(D_{ij}^{*v} - \overline{\rho} \epsilon_{ij}^* + \Phi_{ij})$ . In order to achieve this incompressible limit correctly, a re-arranging of the terms in  $(D_{ij}^{*v} - \overline{\rho} \epsilon_{ij}^* + \Phi_{ij})$  is necessary. If viscous diffusion and dissipation in compressible flows are again defined similarly to their incompressible counterparts, or

$$D_{ij}^{*v} = \frac{\partial}{\partial x_k} \left( \overline{\mu} \frac{\partial \overline{u_i' u_j'}}{\partial x_k} \right) , \quad (14)$$

$$\varepsilon_{ij} = 2\bar{\nu} \overline{\frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_k}}, \quad (15)$$

then the terms  $(D_{ij}^{*\nu} - \bar{\rho}\varepsilon_{ij}^* + \Phi_{ij})$  can be re-arranged to give

$$D_{ij}^{*\nu} - \bar{\rho}\varepsilon_{ij}^* + \Phi_{ij} = D_{ij}^\nu - \bar{\rho}\varepsilon_{ij} - \bar{\rho}\varepsilon_{ij}^c + \Phi_{ij}^* \quad (16)$$

$$\text{where } \varepsilon_{ij}^c = \frac{\bar{\nu}}{3} \left( \overline{\frac{\partial u_i}{\partial x_j} \frac{\partial u_k}{\partial x_k}} + \overline{\frac{\partial u_j}{\partial x_i} \frac{\partial u_k}{\partial x_k}} \right), \quad (17a)$$

$$\begin{aligned} \Phi_{ij}^* = & - \left[ \overline{u_i \frac{\partial p'}{\partial x_j}} + \overline{u_j \frac{\partial p'}{\partial x_i}} \right] + \frac{\bar{\mu}}{3} \left[ \frac{\partial}{\partial x_j} \left( \overline{u_i \frac{\partial u_k}{\partial x_k}} \right) + \frac{\partial}{\partial x_i} \left( \overline{u_j \frac{\partial u_k}{\partial x_k}} \right) \right] \\ & + \frac{\partial \bar{\mu}}{\partial x_k} \left( \overline{u_i \frac{\partial u_k}{\partial x_j}} + \overline{u_j \frac{\partial u_k}{\partial x_i}} \right) - \frac{2}{3} \left[ \frac{\partial \bar{\mu}}{\partial x_j} \overline{u_i \frac{\partial u_k}{\partial x_k}} + \frac{\partial \bar{\mu}}{\partial x_i} \overline{u_j \frac{\partial u_k}{\partial x_k}} \right]. \end{aligned} \quad (17b)$$

Note that (16) reduces to its incompressible counterpart exactly when constant fluid properties are assumed. For compressible flows, an extra term  $\bar{\rho}\varepsilon_{ij}^c$  appears in (16). In addition, three additional terms are found in  $\Phi_{ij}^*$ . The term  $\bar{\rho}\varepsilon_{ij}^c$  is a dilatational term and could be interpreted as compressible or dilatational dissipation. This term is only important for compressible flows.

It should be pointed out that  $\Phi_{ij}^*$  is given by (17b) and, as a result of this particular partitioning, there are several extra terms resulted from compressibility and variable viscosity. However, at high Reynolds number, dimensional arguments reveal that these extra contributions are not important. If pressure diffusion is further neglected, then  $D_{ij}^T$ ,  $\bar{\rho}\varepsilon_{ij}$  and  $\Phi_{ij}^*$  would assume the same form as their incompressible counterparts. Therefore, the high-Reynolds-number incompressible models proposed for these terms<sup>16,32</sup> could be straight-forwardly extended to compressible flows. However, a model for the compressible dissipation term  $\bar{\rho}\varepsilon_{ij}^c$  is required to complete closure. For high-Reynolds-number flows, this compressible dissipation could be assumed to be isotropic. As a result, the following model is proposed:

$$\epsilon_{ij}^c = \frac{2}{3} \delta_{ij} \epsilon^c \quad (18)$$

$$\text{where } \epsilon^c = \frac{\bar{\nu}}{3} \overline{\left( \frac{\partial u_k}{\partial x_k} \right)^2}.$$

The modeling of  $\epsilon^c$  has been attempted by Sarkar et al.<sup>28</sup> They are the first to realize that the contribution of the dilatational dissipation term is important for supersonic shear flows. A simple algebraic model, which is based on an asymptotic analysis and a direct numerical simulation of the simplified governing equations, has been proposed for  $\epsilon^c$ . Their proposal could be modified to become

$$\epsilon^c = \alpha_1 M_t^2 \bar{\epsilon}, \quad (19)$$

where  $\alpha_1$  is a model constant,  $M_t^2 = 2k/\bar{c}^2$ ,  $\bar{\epsilon} = \bar{\mu} \overline{\left( \frac{\partial u_i}{\partial x_i} \right)^2}$  is the dissipation of  $k$  and  $\bar{c}$  is the local mean speed of sound. Therefore,  $M_t$  is the local turbulent Mach number. It should be pointed out that Sarkar et al.'s<sup>28</sup> definition of  $\epsilon^c$  is four times larger than the definition given in (19) as a result of a different splitting of the terms in (16). Consequently,  $\alpha_1$  should take on a value equal to 1/4 of that suggested in Ref. 28. Based on an analysis of decay of compressible isotropic turbulence, Sarkar et al.<sup>28</sup> suggested a value of one for their constant. In other words,  $\alpha_1 = 0.25$ . If  $\alpha_1$  is evaluated based on compressible shear flows, its value would be 0.15. The present study adopts  $\alpha_1 = 0.15$  for the analyses of boundary-layer flows.

The  $k$ -equation is obtained by contracting (12) and making use of (16) and (18) to simplify the resulting equation which can be written as:

$$\frac{D(\bar{\rho}k)}{Dt} = D_k^v + D_k^T + \frac{1}{2}P_{ii} + \frac{1}{2}\Phi_{ii}^* - \frac{1}{2}\bar{\rho}\epsilon_{ii} - \frac{1}{2}\bar{\rho}\epsilon_{ii}^c + \frac{1}{2}G_{ii} + \frac{1}{2}T_{ii}. \quad (20)$$

It can be seen that the terms,  $D_k^T$ ,  $\Phi_{ii}^*$  and  $\bar{\rho}\epsilon_{ii}$ , and the coefficient,  $\bar{u}_i^*$ , appearing in  $G_{ii}$  and  $T_{ii}$  require modeling. Furthermore, when  $\rho$  is assumed to be constant and  $\bar{u}_i^* = 0$ , the last three terms

in (20) are identically zero and the incompressible equation is recovered exactly. The modeling of  $D_k^T$ ,  $\Phi_{ii}^*$  and  $\bar{\rho} \epsilon_{ii}$  could be accomplished by drawing parallels with their incompressible counterparts<sup>26,27</sup>. However, this requires knowledge of their behavior in the near-wall region.

The near-wall behavior of (20) can be analysed by assuming Taylor series expansions about the wall for the fluctuating quantities. This analysis is similar to the incompressible case<sup>26</sup> except that expansions also have to be assumed for  $\rho'$  and  $\theta''$ . The proposed expansions are:

$$\begin{aligned} u'' &= a_1 y + a_2 y^2 + \dots, \\ v'' &= b_1 y + b_2 y^2 + \dots, \\ w'' &= c_1 y + c_2 y^2 + \dots, \\ \theta'' &= d_1 y + d_2 y^2 + \dots, \\ \rho' &= e_1 y + e_2 y^2 + \dots \end{aligned} \tag{21}$$

It should be cautioned that, although the velocity and temperature expansions are physically correct, the expansion for density is an assumption. As pointed out by Bradshaw<sup>11</sup>, the fluctuating temperature and density could not go to zero simultaneously at the wall. Otherwise, it would lead to a zero wall  $p'$ . In general, temperature fluctuation is assumed to be zero at the wall, while  $\rho'$  is not. Here, the assumption is made that  $\rho'$  also goes to zero at the wall, however, its value away from the wall is finite. Since  $\rho'$  is taken to be essentially zero over the whole field in Morkovin's hypothesis<sup>10</sup>, the present approach could be viewed as a partial relaxation of that assumption. Consequently, the proposed model would not be valid for all free-stream Mach number and wall thermal boundary conditions. Therefore, one of the present objective is to analyse the validity and extent of the proposed two-equation model.

For incompressible flows,  $b_1 = 0$  is obtained by imposing the incompressibility condition and becomes a crucial condition in near-wall analysis. This important condition holds the key to the present extension of the near-wall incompressible models to compressible flows. In order to

show that  $b_1$  indeed vanishes under these conditions, the continuity equation for  $\rho'$  is first derived, or

$$\frac{\partial \rho'}{\partial t} + \frac{\partial}{\partial x_k} \left( \bar{\rho} u_k' + \rho' \langle U_k \rangle + \rho' u_k' \right) = 0. \quad (22)$$

Expansions (21) are then substituted into the above equation. If  $\langle U_k \rangle = 0$  at the wall is used, it can be easily verified that, under the assumption of (21),  $b_1 = 0$  is still a valid condition for compressible flows, irrespective of the thermal boundary condition. Therefore, the assumed  $\rho'$  expansion facilitates the modeling of compressible flows, because all terms in (20) have similar forms as their incompressible counterparts except the extra  $\epsilon_{ii}^c$  term which needs to be analyzed.

Using definition (18) for  $\epsilon_{ij}^c$ , it is easily verified that  $\epsilon_{ii}^c$  is of order  $y^2$ . The high-Reynolds-number model (19) also has similar behavior near a wall. Therefore, it is proposed to extend (19) to near-wall flow without modification, while the near-wall balance provided by the exact  $\epsilon_{ii}^c$  is taken into consideration by combining it with the  $\Phi_{ii}^*$  term. As for  $\bar{\rho} \epsilon_{ii}$ , it could be modeled by following the arguments presented in Refs. 26 and 27 for incompressible flows. In essence, Refs. 26 and 27 argue that the incompressible  $\epsilon_{ii}$  can be set equal to  $2\epsilon$  and the near-wall corrections proposed for  $\epsilon_{ij}$  have little or no effects on the behavior of  $\epsilon_{ii}$  in the region near a wall. This means that  $\epsilon_{ii}$  in the near-wall region as well as the region away from the wall can be approximated by  $2\epsilon$ . In view of this, the model for  $\bar{\rho} \epsilon_{ii}$  can be assumed to be given by  $2\bar{\rho} \epsilon$ . Based on this model, equation (15) and expansions (21), it can be easily shown that the leading term of  $\epsilon$  in the near-wall region is a constant equal to its wall value  $\epsilon_w$ . Again, the behavior is similar to the incompressible case.

Near-wall analysis again shows that turbulent diffusion is a higher order term and its high-Reynolds-number model could be adopted because it does not affect near-wall balance of the  $k$ -equation. Consistent with the assumption of gradient transport for two-equation models, the incompressible model for turbulent diffusion of  $k$  is extended to compressible flows by writing  $D_k^T$



$= \partial((\bar{\mu}_t/\sigma_k)\partial k/\partial x_i)/\partial x_i$ , where  $\sigma_k$  is a constant and  $\bar{\mu}_t$  is the turbulent viscosity defined by  $\bar{\mu}_t = C_\mu f_\mu \bar{\rho} k^2/\epsilon$ . In this definition,  $C_\mu$  is a model constant while  $f_\mu$  is a damping function to be defined later. Based on (21), the leading order term of  $k$  in the near-wall region is  $y^2$ . Since  $\epsilon = \epsilon_w$  in this region,  $k^2/\epsilon$  has to be of order  $y^4$ . If the shear stress is defined with respect to  $\bar{\mu}_t$ , then it can be shown that the leading order term of the shear stress has to be of order  $y^3$  in the near-wall region. Therefore, it follows that  $\bar{v}_t = \bar{\mu}_t/\bar{\rho}$  is also of order  $y^3$  near a wall and this, in turn, leads to a similar behavior for the modeled  $D_k^T$  term in the near-wall region. This behavior is consistent with the behavior of the exact term  $D_k^T$  appearing in (20). In other words, the modeled  $D_k^T$  does not affect the near-wall balance of (20).

According to (18),  $\epsilon_{ii}^c = 2\epsilon^c$ . As such, the near-wall behavior of the exact  $\epsilon_{ii}^c$  is not properly accounted for by the proposed model. In the above discussion, it is argued that the near-wall behavior of  $\epsilon_{ii}^c$  could be modeled together with the term  $\Phi_{ii}^*$ . In order to analyse the near-wall behavior of the combined term  $(\Phi_{ii}^* + \bar{\rho}\epsilon_{ii}^c)$ , the behavior of  $G_{ii}$  and  $T_{ii}$  near a wall has to be studied. The appearance of mean pressure in  $G_{ii}$  makes the analysis slightly more difficult. However, the difficulty could be circumvented by making use of the mean momentum equation (10). The final analysis shows that the combined  $(G_{ii} + T_{ii})$  term has the following near-wall behavior; namely,

$$G_{11} + T_{11} \rightarrow O(y^2); G_{33} + T_{33} \rightarrow O(y^2); G_{22} + T_{22} \rightarrow O(y^3) \quad . \quad (23)$$

This means that, to the lowest order, the near-wall behavior of  $(\Phi_{ii}^* + \bar{\rho}\epsilon_{ii}^c)$  is similar to its incompressible counterpart<sup>26</sup>. For incompressible flows, the term,  $\Phi_{ii}^*$ , can be written into a pressure diffusion part and a pressure redistribution part. Pressure redistribution is identically zero and since pressure diffusion is relatively small, it is usually neglected. Such is not the case for compressible flows. The term,  $\Phi_{ii}^*$ , can again be partitioned into a pressure diffusion part, which could be neglected, and a term involving pressure-velocity-gradient correlation. This latter term does not vanish because fluid volume changes as a result of density variation. Therefore, an

argument could be made to model the term,  $(\Phi_{ii}^* + \bar{\rho} \epsilon_{ii}^c)$ , to account for dilatational effects only. In view of this, the following model is proposed, or

$$(\Phi_{ii}^* + \bar{\rho} \epsilon_{ii}^c) = -\gamma \bar{\rho} k \left( \frac{\partial \langle U_i \rangle}{\partial x_i} \right), \quad (24)$$

where  $\gamma$  is a model constant.

The proposed models still fail to close the  $k$ -equation because of the presence of  $\overline{u_i''}$  in  $T_{ij}$  and  $G_{ij}$ . Therefore, it is necessary to shed some light on the modeling of  $\overline{u_i''}$ , which is identically zero for incompressible flows. Using Favre averaging, it can be shown that  $\overline{-\rho' u_i''} = \bar{\rho} \overline{u_i''}$ . In other words,  $\overline{u_i''} = -\overline{\rho' u_i''} / \bar{\rho}$ . Previous proposals for  $\overline{-\rho' u_i''}$  are based on the gradient transport assumption; namely,

$$\overline{-\rho' u_i''} = \frac{\bar{v}_i}{\sigma_p} \frac{\partial \bar{\rho}}{\partial x_i}, \quad (25)$$

where  $\sigma_p$  is a model constant. However, a more elaborate way to model the term is to adopt the proposal,

$$\overline{u_i''} = -\frac{\overline{\rho' u_i''}}{\bar{\rho}} = C_p \frac{k}{\rho \epsilon} \langle u_i'' u_j'' \rangle \frac{\partial \bar{\rho}}{\partial x_j}, \quad (26)$$

where  $C_p$  is a model constant. Alternatively, the term can also be modeled by

$$\overline{u_i''} = \beta \frac{\overline{u_i'' \theta''}}{\langle \theta \rangle}, \quad \text{with } \beta = - \left( \frac{\partial \bar{\rho}}{\partial \langle \theta \rangle} \right)_p \frac{\langle \theta \rangle}{\bar{\rho}}, \quad (27)$$

where  $\beta$  equals to unity for an ideal gas.

The near-wall behavior of the modeled  $k$ -equation can now be analysed using expansions (21). It can be easily shown that in the region very near a wall, the modeled  $k$ -equation is in

balance up to order  $y$ . Consequently, it does not need further modifications to achieve a consistent asymptotic behavior near a wall.

## 5. Modelling of the Dissipation-Rate Equation

The exact transport equation for the solenoidal dissipation rate ( $\overline{\rho} \epsilon$ ) can be similarly derived as in the Reynolds-stress equation (12). If the approach of the previous section is adopted, the various terms in the equation can be grouped into terms that are essentially similar to their incompressible counterparts<sup>16</sup>. Therefore, this derivation will give an  $\epsilon$ -equation similar to the incompressible equation but with three additional terms involving  $\overline{u_i}$  and  $\overline{\mu}$ . If these three terms are denoted by  $S_{\epsilon 1}$ ,  $S_{\epsilon 2}$  and  $S_{\epsilon 3}$ , the exact compressible equation for  $\overline{\rho} \epsilon$  can be written as:

$$\begin{aligned} \frac{D\overline{\rho}\epsilon}{Dt} = & \overline{v} \frac{\partial}{\partial x_k} \left( \frac{\partial \overline{\rho}\epsilon}{\partial x_k} \right) - \frac{\partial}{\partial x_k} \left[ \overline{\rho} \langle \overline{u_k} \epsilon \rangle + 2\overline{v} \frac{\partial \overline{p'}}{\partial x_i} \frac{\partial \overline{u_k}}{\partial x_i} \right] \\ & - 2\overline{\rho} \overline{v} \frac{\partial \langle U_i \rangle}{\partial x_k} \left[ \left\langle \frac{\partial \overline{u_i}}{\partial x_j} \frac{\partial \overline{u_k}}{\partial x_j} \right\rangle + \left\langle \frac{\partial \overline{u_i}}{\partial x_i} \frac{\partial \overline{u_k}}{\partial x_k} \right\rangle \right] \\ & - 2\overline{\rho} \overline{v} \langle \overline{u_k} \frac{\partial \overline{u_i}}{\partial x_j} \rangle \frac{\partial^2 \langle U_i \rangle}{\partial x_k \partial x_j} - 2\overline{\rho} \overline{v} \langle \frac{\partial \overline{u_i}}{\partial x_k} \frac{\partial \overline{u_i}}{\partial x_j} \frac{\partial \overline{u_k}}{\partial x_j} \rangle \\ & - 2\overline{v}^2 \frac{\partial^2 \overline{u_i}}{\partial x_k \partial x_j} \frac{\partial^2 \overline{u_i}}{\partial x_k \partial x_j} + S_{\epsilon 1} + S_{\epsilon 2} + S_{\epsilon 3} . \end{aligned} \quad (28)$$

The term  $S_{\epsilon 3}$  is too cumbersome to write out in full. It is related to the gradient of  $\overline{v}$  and  $\overline{u_i}$  and is very much like the extra terms in (17). On the other hand, the terms  $S_{\epsilon 1}$  and  $S_{\epsilon 2}$  are given by

$$S_{\epsilon 1} = -2\overline{v} \frac{\partial \overline{u_i}}{\partial x_k} \frac{\partial^2 \overline{p}}{\partial x_k \partial x_i} \quad \text{and} \quad S_{\epsilon 2} = 2\overline{v} \frac{\partial \overline{u_i}}{\partial x_k} \frac{\partial^2 \langle \tau_{ij} \rangle}{\partial x_k \partial x_j} . \quad (29)$$

These terms are not only related to the gradient of  $\overline{u_i}$  but they are also functions of the gradients of mean pressure and mean viscous stresses. They are like the terms  $G_{ii}$  and  $T_{ii}$  in the  $k$ -equation and are explicitly dependent on compressibility effects. Therefore,  $S_{\epsilon 1}$  and  $S_{\epsilon 2}$  have to be modeled

separately<sup>32</sup> just as in the case of the  $k$ -equation. The term  $S_{\epsilon 3}$  can be grouped with the other terms that need modeling and treated in a manner similar to that in the  $k$ -equation.

It has been pointed out that the  $\epsilon$ -equation is the most difficult to model even for incompressible flows<sup>13,16,17,26,27,32</sup>. The reason being that many of the terms in the exact equation are either not known or could not be measured accurately at present. Consequently, the incompressible  $\epsilon$ -equation is modeled in an ad hoc manner to resemble the  $k$ -equation in form so that the right hand side of the  $\epsilon$ -equation again consists of four terms; namely, viscous diffusion, turbulent diffusion, production and destruction of  $\epsilon$ . The equation is further modified for near-wall flows by adding an extra destruction term  $\xi$  so that the modeled equation remains balance as a wall is approached. There is a lack of measurements in compressible flows, therefore, a rigorous modeling of the compressible  $\epsilon$ -equation is not possible at present. An alternative is to extend the high-Reynolds-number incompressible models to compressible flows and then seek a near-wall correction to the modeled  $\epsilon$ -equation along the line suggested in Ref. 27. In view of this, (28) is not a convenient form to work with. The proposal by Speziale and Sarkar<sup>13</sup> with the dilatational effects written out explicitly will be more appropriate.

Following Speziale and Sarkar<sup>13</sup>, the modeled transport equation for  $\epsilon$  with a near-wall correction is written in the simplified form; namely

$$\frac{D\bar{\rho}\epsilon}{Dt} = \frac{\partial}{\partial x_i} \left( \bar{\mu} \frac{\partial \epsilon}{\partial x_i} \right) + D_{\epsilon}^t + P_{\epsilon} - \Delta_{\epsilon} - \frac{4\bar{\rho}}{3} \epsilon \frac{\partial \langle U_i \rangle}{\partial x_i} + \xi, \quad (30)$$

where  $D_{\epsilon}^t$  is the turbulent transport of  $\epsilon$ ,  $P_{\epsilon}$  is the production of  $\epsilon$  due to deviatoric strains,  $\Delta_{\epsilon}$  is the destruction of  $\epsilon$  and  $\xi$  is a near-wall correction for compressible flows. The second last term on the right hand side of (30) is exact and results from the writing of (28) into the form of (30). When the dissipation-rate equation is formulated in this form, it is reducible exactly to its incompressible counterpart and, therefore, the terms  $D_{\epsilon}^t$ ,  $P_{\epsilon}$  and  $\Delta_{\epsilon}$  can be modeled by a variable

density extension of their incompressible models. Following the suggestions of Refs. 13 and 27, the models proposed for  $D_\epsilon^t$ ,  $P_\epsilon$  and  $\Delta_\epsilon$  are:

$$D_\epsilon^t = \frac{\partial}{\partial x_i} \left( \frac{\overline{\mu_t} \partial \epsilon}{\sigma_\epsilon \partial x_i} \right), \quad (31a)$$

$$P_\epsilon = - C_{\epsilon 1} \overline{\rho} \frac{\epsilon}{k} \left( \overline{u_i u_j} \right) \left( \frac{\partial \langle U_i \rangle}{\partial x_j} - \frac{1}{3} \frac{\partial \langle U_k \rangle}{\partial x_k} \delta_{ij} \right), \quad (31b)$$

$$\Delta_\epsilon = C_{\epsilon 2} \overline{\rho} \frac{\epsilon \tilde{\epsilon}}{k}, \quad (31c)$$

where the model constants  $C_{\epsilon 1}$  and  $C_{\epsilon 2}$  are the same as those given in Ref. 27 for incompressible flows and  $\tilde{\epsilon} = \epsilon - \epsilon_w$ . It should be noted that the mean dilatational effects are accounted for exactly by (31b) for compressible flows and that these models reduce exactly to their incompressible counterparts when the flow Mach number becomes very small. In addition, the ordering of these model terms is similar to their incompressible counterparts. Therefore, the near-wall function  $\xi$  can be determined in a manner similar to that used in Ref. 27.

The incompressible form of (30) with model terms given in (31) is identical to that proposed in Refs. 26 and 27. In these studies, the coincidence condition suggested by Shima<sup>21</sup> was used to determine  $\xi$ . This is equivalent to requiring the modeled  $\epsilon$ -equation to achieve balance behavior in the near-wall region at least up to order  $y$ . The approach used to deduce  $\xi$  is to assumed a functional form for  $\xi$  with two undetermined model constants. One of the constant can be determined from near-wall analysis, while the other is evaluated using computer optimization. The  $\xi$  function thus determined has been used in Ref. 27 to calculate flat plate boundary-layer flows and in Ref. 26 to calculate fully-developed channel and pipe flows. These calculations were carried out over a wide range of flow Reynolds number. The results were compared with direct simulation data as well as measurements. Very good agreement has been found for both the limiting behavior of the turbulence quantities and  $\epsilon$  when compared to direct simulation data<sup>33-35</sup>.

Furthermore, the two-equation model calculations of Ref. 27 are found to give better results than those obtained by Speziale et al.<sup>36</sup> In view of this success, the same approach can be used to determine  $\xi$  for compressible flows.

The functional form assumed in Ref. 27 is adopted here, or

$$\xi = f_{w,2} \bar{\rho} \left[ -N \frac{\varepsilon \tilde{\varepsilon}}{k} + M \frac{\varepsilon^{*2}}{k} \right], \quad (32)$$

where  $f_{w,2}$  is a damping function that goes to one at the wall and zero far away from the wall. It is defined in Ref. 27 as  $f_{w,2} = e^{-(R_t/64)^2}$ , where  $R_t = k^2/\sqrt{\nu} \varepsilon$  is the turbulent Reynolds number. The function  $\varepsilon^*$  is defined as  $\varepsilon^* = \varepsilon - 2\sqrt{\nu} k/y^2$  by generalizing the incompressible definition used in Ref. 27. Similarly,  $\tilde{\varepsilon}$  is defined with  $\varepsilon_w$  specified by  $\varepsilon_w = 2\sqrt{\nu} (\partial \sqrt{k}/\partial x_j)_w^2$ . Once  $\xi$  is postulated, the near-wall behavior of (30) and the modeled terms of (31) can be analysed using expansions (21). If the modeled equation is again required to be in balance up to order  $y$ , then it can be easily shown that  $N = 2 - C_{\varepsilon 2}$  because the mean dilatational terms are of order  $y$ . Therefore, to order  $y^0$  they do not contribute to  $\xi$ . In Ref. 27, the part involving  $C_{\varepsilon 2}$  in  $N$  is grouped together with  $M$  to give  $M_1 = (C_{\varepsilon 2} \varepsilon \tilde{\varepsilon}/\varepsilon^{*2} + M)$  and its value is determined through computer optimization studies. Again, the same procedure is followed in the present study to determine  $M_1$ .

Finally, to complete closure of the governing equations, gradient transport is assumed for the Reynolds stresses and the relation is given by

$$-\overline{\rho \langle u_i' u_j' \rangle} = \bar{\mu}_t \left[ \frac{\partial \langle U_i \rangle}{\partial x_j} + \frac{\partial \langle U_j \rangle}{\partial x_i} - \frac{2}{3} \delta_{ij} \frac{\partial \langle U_m \rangle}{\partial x_m} \right] - \frac{2}{3} \delta_{ij} \bar{\rho} k. \quad (33)$$

In addition, a constant turbulent Prandtl number is assumed so that heat and momentum diffusivity can be related by  $Pr_t = \sqrt{\nu} \bar{\alpha}_t$  where  $\bar{\alpha}_t$  is the turbulent thermal diffusivity. The damping function  $f_\mu$  appearing in the definition of  $\bar{\mu}_t$  can now be defined. In view of the similarity of the present  $k$

and  $\epsilon$  equations with those for incompressible flows, the  $f_\mu$  used in Ref. 27 is adopted here. It is defined as

$$f_\mu = (1 + 3.45/\sqrt{R_t}) \tanh(y^+/115) \quad , \quad (34)$$

where  $y^+ = yu_\tau/\sqrt{\nu}$  is the wall coordinate and  $u_\tau(x)$  is the friction velocity. In this definition,  $y$  is taken to be the normal coordinate and  $x$  the stream coordinate.



## 6 . Model Validation

A first step to validate the two-equation model for compressible flows is to apply it to calculate flat plate boundary-layer flows with different wall boundary conditions and free-stream Mach numbers. In this initial attempt, heat flux is not modeled separately. Instead, it is related to momentum flux via the assumption of a constant turbulent Prandtl number. The rationale for doing this is to carefully assess the assumption made in (21) concerning the expansion for  $\rho'$ , whose validity affects the near-wall analysis used to justify the direct extension of the incompressible near-wall function  $\xi$  to compressible flows. By choosing the simplest type of compressible flows to validate the model, a careful analysis of the validity and extent of the expansion for  $\rho'$  can be carried out. A second objective of this validation is to determine, if possible, the validity and extent of Morkovin's hypothesis. In other words, it is hoped to evaluate the Mach number range and the type of wall thermal boundary conditions in which the effects of fluctuating density can be neglected in the modeling of the governing equations. Therefore, the experimental data chosen for comparisons are selected from three different groups; one with adiabatic wall boundary condition and varying free-stream Mach number, another with constant wall temperature and varying free-stream Mach number, and finally the variations of skin friction with free-stream Mach number and wall temperature as prescribed by the van Driest II formulae given in Ref. 9. The data for the first two groups are selected from Ref. 31. They are cases 55010504, 53011302 and 73050504 with adiabatic wall boundary condition and case 59020105 with constant wall temperature. The free-stream Mach number ( $M_\infty$ ) for these cases are 2.244, 4.544, 10.31 and 5.29, respectively. Only one case with constant wall temperature is selected. The reasons being that the other cases reported in Ref. 31 are either not accurate as far as the measured skin friction is concerned or the measurements of the mean velocity and temperature are doubtful.

In the following, the governing equations for compressible flat plate boundary-layer flows are first presented. Then the calculations and comparisons with data are examined and the validity

of the two-equation model is studied in detail. Finally, the model performance and its proposed improvement are discussed.

### 6.1 Governing Equations

Two-dimensional, steady compressible boundary layers on a flat plate are considered. If the usual boundary-layer approximations are made, then governing equations (9) - (11), (20) and (30) can be substantially simplified. For the sake of completeness, the boundary-layer equations in Cartesian x-y coordinates are listed here as:

$$\frac{\partial}{\partial x}(\bar{\rho} \langle U \rangle) + \frac{\partial}{\partial y}(\bar{\rho} \langle V \rangle) = 0, \quad (35)$$

$$\bar{\rho} \langle U \rangle \frac{\partial \langle U \rangle}{\partial x} + \bar{\rho} \langle V \rangle \frac{\partial \langle U \rangle}{\partial y} = \frac{\partial}{\partial y} \left[ (\bar{\mu} + \bar{\mu}_t) \frac{\partial \langle U \rangle}{\partial y} \right], \quad (36)$$

$$\begin{aligned} \bar{\rho} \langle U \rangle \frac{\partial \langle H \rangle}{\partial x} + \bar{\rho} \langle V \rangle \frac{\partial \langle H \rangle}{\partial y} = \\ \frac{\partial}{\partial y} \left[ \left( \frac{\bar{\mu}}{\text{Pr}} + \frac{\bar{\mu}_t}{\text{Pr}_t} \right) \frac{\partial \langle H \rangle}{\partial y} + \bar{\mu} \left( 1 - \frac{1}{\text{Pr}} \right) \left( 1 + \frac{\text{Pr}(1 - \text{Pr}_t) \bar{\mu}_t}{\text{Pr}_t(1 - \text{Pr}) \bar{\mu}} \right) \langle U \rangle \frac{\partial \langle U \rangle}{\partial y} - \left( \frac{\bar{\mu}}{\text{Pr}} + \frac{\bar{\mu}_t}{\text{Pr}_t} \right) \frac{\partial k}{\partial y} \right], \end{aligned} \quad (37)$$

$$\begin{aligned} \bar{\rho} \langle U \rangle \frac{\partial k}{\partial x} + \bar{\rho} \langle V \rangle \frac{\partial k}{\partial y} = \frac{\partial}{\partial y} \left[ \left( \bar{\mu} + \frac{\bar{\mu}_t}{\sigma_k} \right) \frac{\partial k}{\partial y} \right] + \bar{\mu}_t \left( \frac{\partial \langle U \rangle}{\partial y} \right)^2 - \frac{2}{3} \bar{\mu}_t \left( \frac{\partial \langle U \rangle}{\partial x} + \frac{\partial \langle V \rangle}{\partial y} \right)^2 \\ - \frac{2}{3} \bar{\rho} k \left( \frac{\partial \langle U \rangle}{\partial x} + \frac{\partial \langle V \rangle}{\partial y} \right) - \bar{\rho}(\epsilon + \epsilon^c) - \gamma \bar{\rho} k \left( \frac{\partial \langle U \rangle}{\partial x} + \frac{\partial \langle V \rangle}{\partial y} \right) + \bar{u}'' \frac{\partial}{\partial y} \left( \bar{\mu} \frac{\partial \langle U \rangle}{\partial y} \right), \end{aligned} \quad (38)$$

$$\begin{aligned} \bar{\rho} \langle U \rangle \frac{\partial \epsilon}{\partial x} + \bar{\rho} \langle V \rangle \frac{\partial \epsilon}{\partial y} = \frac{\partial}{\partial y} \left[ \left( \bar{\mu} + \frac{\bar{\mu}_t}{\sigma_\epsilon} \right) \frac{\partial \epsilon}{\partial y} \right] + C_{\epsilon 1} \frac{\epsilon}{k} \bar{\mu}_t \left( \frac{\partial \langle U \rangle}{\partial y} \right)^2 \\ - C_{\epsilon 2} \bar{\rho} \frac{\epsilon \tilde{\epsilon}}{k} - \frac{4}{3} \bar{\rho} \epsilon \frac{\partial \langle U \rangle}{\partial y} + \xi. \end{aligned} \quad (39)$$

In writing down these equations, (33) has been used to relate the shear stress to the mean velocity gradient. It is also noted that  $G_{ij}$  is zero for flat plate boundary-layer flows while the only term of importance in  $T_{ij}$  is that given by  $\bar{u}''$ . If (25) is used to evaluate  $\bar{u}''$ , then  $T_{ij}$  is of order  $y^2$  and is

not important in the near-wall region, which according to So et al.<sup>27</sup> is bounded by  $0 \leq y^+ \leq 5$ . The model constants are taken from Refs. 27 and 32 and are given by:  $C_\mu = 0.096$ ,  $C_{\epsilon 1} = 1.5$ ,  $C_{\epsilon 2} = 1.83$ ,  $\sigma_k = 0.75$ ,  $\sigma_\epsilon = 1.45$ ,  $\alpha_1 = 0.15$ ,  $\sigma_\rho = 0.5$  and  $\gamma = 0.182$ . The turbulent Prandtl number  $Pr_t$  is specified as 0.9, while  $Pr$  is assumed constant and taken to be 0.74. Sutherland and power laws are used to evaluate fluid viscosity at an appropriate reference temperature. The constants  $\sigma_\rho$  and  $\alpha_1$  are determined by calibrating the calculated results against some classic experiments such as those provided in Refs. 29 and 30. Once determined, they are kept constant for all other calculations. In fact, the cases to be compared in the next section are different from the cases used to calibrate  $\sigma_\rho$  and  $\alpha_1$ . As for  $N$  and  $M_1$ , the final choice of values adopted are 2 and 1.5, respectively, just as in the case of incompressible flows.

The boundary conditions are no slip at the wall for the mean velocities and  $k$ , and zero heat flux or constant wall temperature at the wall for the enthalpy. As for the dissipation rate, its value at the wall is given by  $2\bar{v}(\partial\sqrt{k}/\partial y)_w^2$ . At the edge of the boundary layer, the free-stream conditions are specified for both the mean stream velocity and the enthalpy. In principle, the turbulence quantities,  $k$  and  $\epsilon$ , should be zero in the free stream. However, in practice, they are assumed to take on some very small values, of the order of  $10^{-7}$ , in the free stream. Thus formulated, the above equations and the appropriate boundary conditions can be solved numerically using the boundary-layer code developed by Anderson and Lewis<sup>30</sup> and modified by So et al.<sup>27</sup>.

All measurements used to validate the near-wall two-equation model are drawn from Ref. 31. The calculations are carried out over the range,  $0 < M_\infty < 10$ , for adiabatic wall boundary condition and over the range,  $0 < \Theta_w/\Theta_r < 1$ , for constant wall temperature condition. Here,  $\Theta_w$  is the wall temperature and  $\Theta_r$  is the recovery temperature. It should be pointed out that  $\Theta_w/\Theta_r = 1.0$  corresponds to adiabatic wall boundary condition while  $\Theta_w/\Theta_r < 1$  indicates that the wall is cooled. Since only mean flow properties are available from Ref. 31, comparisons are made with these measured quantities and another set of calculations using the  $k-\omega$  model of Wilcox<sup>8</sup>. All comparisons are made at the same momentum thickness Reynolds number ( $R_\theta$ ) as the

measurements. The variations of skin friction with  $M_\infty$  and  $\Theta_w/\Theta_r$  are compared with the van Driest II formulae reported in Ref. 9. In these comparisons, the calculations are carried out at a fixed  $R_\theta$  as specified in Ref. 9. Finally, an assessment of the effects of compressibility on near-wall flows is attempted by comparing the near-wall behavior of the calculated turbulence statistics for the different cases investigated.

## 6.2 Comparisons with Data

The results are organized in the following manner for presentation. Mean stream velocities are normalized by  $u_\tau$  to give  $u^+$  and they are plotted versus  $\ln y_w^+$  (Fig. 1), where  $y_w^+$  is defined as  $u_\tau y / \sqrt{\nu_w}$ . On the other hand, mean temperatures are normalized by  $\Theta_\infty$ , the free-stream temperature, and are shown versus  $y/\delta$  (Fig. 2), where  $\delta$  is the boundary layer thickness defined as the location of  $y$  in which  $\langle U \rangle / U_\infty = 1.0$  as specified by the measurements. Here,  $U_\infty$  is the free-stream velocity. Plots for the turbulence properties,  $k^+ = k/u_\tau^2$ ,  $\epsilon^+ = \epsilon \bar{\nu} / u_\tau^4$ ,  $-\overline{uv}^+ = -\overline{uv}/u_\tau^2$  and  $-\overline{\theta v}^+ = -\overline{\theta v} / U_\infty \Theta_\infty$ , are presented in terms of  $y_w^+$  and  $y/\delta$ . Near-wall behavior of  $k^+$ ,  $\epsilon^+$ ,  $-\overline{uv}^+$  and  $-\overline{\theta v}^+$  (Figs. 3 - 6) are discussed first, then the distributions of  $k^+$  and  $-\overline{uv}^+$  versus  $y/\delta$  (Figs. 7 and 8) are examined. In Figs 7 and 8,  $\delta$  is not interpreted from measurements; rather it is evaluated at the  $y$  location where  $\langle U \rangle / U_\infty = 0.9974$ . Only the budget of  $k$  in the near-wall region for case 73050504 ( $M_\infty = 10.31$  and  $\Theta_w/\Theta_r = 1.0$ ) is presented (Fig. 9) because the  $k$  budgets for the other cases are essentially similar to that for case 73050504. The effects of Mach number on the asymptotic behavior of  $k$  are examined by plotting  $a_k$  versus  $M_\infty$  (Fig. 10), where  $a_k$  is the leading coefficient in the expansion of  $k^+$  in terms of  $y_w^+$ . According to Ref. 27,  $\epsilon_w^+ = 2a_k$ . Therefore, by examining the behavior of  $a_k$  versus  $M_\infty$ , the variation of  $\epsilon_w^+$  with Mach number is also evident. Other asymptotic properties are tabulated in Table 1 for comparison. The effects of Reynolds number on the calculated results are also investigated for the cases 53011302 and 59020105 (Figs. 11 and 12). Finally, the effects of  $M_\infty$  on the skin friction coefficient,  $C_f = 2\tau_w / \rho_\infty U_\infty^2$ , is compared in Fig. 13 with the van Driest II result<sup>9</sup> for adiabatic wall boundary

condition. As for the case of  $C_f$  versus  $\Theta_w/\Theta_f$  for a constant  $M_\infty$ , the results are listed in Table 2 for comparison with the van Driest II data obtained from Ref. 9.

Two versions of the present k- $\epsilon$  model are used to calculate the boundary-layer flows. One designated k- $\epsilon$  model/1 solves the k and  $\epsilon$  transport equations as given in (38) and (39). The second designated k- $\epsilon$  model/2 solves (38) and (39) with all additional compressible terms neglected. In other words, the two-equation model for k- $\epsilon$  model/2 is a direct variable density extension of the two-equation incompressible model of Ref. 27. These calculations can be used to evaluate the validity and extent of Morkovin's hypothesis and the importance of having an asymptotically consistent near-wall correction for two-equation models.

Four sets of  $u^+$  results are shown in Figs. 1a - 1d. Also shown in the figures are the calculated and measured  $C_f$  and the  $C_f$  determined from the van Driest II formula of Ref. 9. The log-law is shown in Fig. 1a to illustrate the log region of the calculated and measured flows. It can be seen that  $C_f$  is predicted correctly by all three models with a maximum error of less than 5%. For the cooled wall case, the measured  $C_f$  is substantially higher than the van Driest II value and, according to Ref. 31, it is not as accurate as the measured  $C_f$  for the other cases studied. The model calculations are in good agreement with the van Driest II value for this case. Calculated  $u^+$  profiles from k- $\epsilon$  model/1 and k- $\epsilon$  model/2 correlate well with measurements, even for the  $M_\infty = 10.31$  case (Fig. 1c). This means that the present k- $\epsilon$  models are quite valid for compressible flows up to  $M_\infty = 10$ . The models represent improvements over existing models whose predictions are correct only for  $M_\infty \leq 5$  (see e.g. Ref. 37). Since most existing models are direct variable density extensions of conventional near-wall two-equation incompressible models and most of these incompressible models do not have an asymptotically consistent near-wall correction, the discrepancies display by these models for  $M_\infty > 5$  is understandable. Present results show that, if the near-wall correction is asymptotically consistent, the incompressible models can be straight-forwardly extended to compressible flows with a free-stream Mach number as high as 10.

It seems that k- $\epsilon$  model/2 gives a better correlation with velocity measurements (Figs. 1c and 1d) and thereby suggests that the additional compressible terms in (38) and (39) are not important. However, an examination of the mean temperature results tends not to support this conclusion. Further evidence that an asymptotically consistent near-wall correction is important can be gleaned from the predictions of the k- $\omega$  model. As  $M_\infty$  increases, the discrepancies between predictions and measurements also increase. For low  $M_\infty$ , the slope of the predicted log-law as calculated by the k- $\omega$  model is about the same as the slope of the log-law shown in Fig. 1a. However, the predicted slope increases as  $M_\infty$  increases. The same is also true for the cooled wall case where the  $u^+$  profile is not shown in Fig. 1d but is shown in Fig. 12a when the effect of Reynolds number are analysed. This behavior is not observed in both the k- $\epsilon$  model/1 and k- $\epsilon$  model/2 predictions, however.

The mean temperature profile comparisons are shown in Figs. 2a - 2d. As expected, predictions by the k- $\omega$  model compare less favorably with measurements. Here again, the predictions of the k- $\omega$  model are shown in Fig. 12b rather than in Fig. 2d. This is even true for low free-stream Mach number (Fig. 2a). As  $M_\infty$  increases, discrepancies between measurements and k- $\omega$  predictions also increase. On the other hand, the agreement between the k- $\epsilon$  model/1 predictions and data improves as  $M_\infty$  increases. This is not true for k- $\epsilon$  model/2 where the disagreement with data is quite substantial at  $M_\infty = 10.31$  (Fig. 2c). The predictions of the cooled wall case (Fig. 2d) tell a different story. It seems that k- $\epsilon$  model/2 gives an overall better prediction of the cooled wall case compared to that of k- $\epsilon$  model/1 (Figs. 1d and 2d). The following three reasons could be put forward to explain this behavior. Firstly, the  $\rho'$  expansion may not be totally valid for constant wall temperature boundary condition. Secondly, the proposed compressible models may be more applicable for adiabatic wall boundary condition. Thirdly, the assumption of a constant turbulent Prandtl number may not be correct.

The near-wall distributions of  $k^+$  for the four cases are shown in Figs. 3a - 3d. Only the predictions of the k- $\epsilon$  model/1 and k- $\omega$  model are compared. The calculations of k- $\epsilon$  model/2 are not shown; instead, the limiting behavior of  $k^+$ ,  $-\overline{uv}^+$  and  $-\overline{\theta v}^+$  is tabulated in Table 1 for comparison. In general, the predictions of the k- $\omega$  model are substantially lowered than those of the k- $\epsilon$  model/1. The peaks are about 40% lower than those predicted by the k- $\epsilon$  model/1 and the locations where the peaks occur are calculated to be further away from the wall than the k- $\epsilon$  model/1 predictions. According to the k- $\epsilon$  model/1 predictions, the peak of  $k^+$  decreases as  $M_\infty$  increases. The decrease is more than 20% over a Mach number range of 10. On the other hand, a slight cooling of the wall at an  $M_\infty$  of 5.29 causes the peak of  $k^+$  to decrease to the same level as that for the case of adiabatic wall with  $M_\infty = 10.31$ . These results suggest that wall cooling has more influence on reducing turbulent mixing compared to compressibility effects. Overall, compressibility reduces turbulent mixing and the reduction increases with  $M_\infty$ . The near-wall distributions of  $\epsilon^+$  as calculated by k- $\epsilon$  model/1 are plotted in Figs. 4a - 4d. It can be seen that the distributions are very similar to those shown in Ref. 27 for incompressible flows. The variations of  $\epsilon_w^+$  with  $M_\infty$  and  $\Theta_w/\Theta_r$  are very similar to those of  $k^+$ . Again, maximum  $\epsilon^+$  is found at the wall and a plateau in  $\epsilon^+$  is found in the range,  $7 \leq y_w^+ \leq 13$ . This means that compressibility has little or no effects on the near-wall behavior of  $\epsilon^+$ . The exception is that increases in compressibility and wall cooling tend to decrease  $\epsilon_w^+$ .

In general, the k- $\omega$  model gives a very accurate prediction of  $-\overline{uv}^+$  near a wall. Its predictions are as good as those given by k- $\epsilon$  model/1 (Figs. 5a - 5d). From this set of predictions, the following observations can be made. Firstly, the peak of  $-\overline{uv}^+$  decreases with increase  $M_\infty$  and decrease  $\Theta_w/\Theta_r$ . Secondly, as  $M_\infty$  increases and  $\Theta_w/\Theta_r$  decreases, the location of the peak moves towards the wall. Thirdly, the rate of decrease of  $-\overline{uv}^+$  in the range,  $30 \leq y_w^+ \leq 100$ , increases as  $M_\infty$  increases. Finally, the asymptotic near-wall behavior of  $-\overline{uv}^+$  is listed in Table 1 for comparison. Much the same behavior is also true for  $-\overline{\theta v}^+$  whose distributions in the near-wall region are shown in Figs. 6a - 6d. If the distributions of  $\overline{\theta v}^+$  are plotted instead of  $-\overline{\theta v}^+$ ,

the curves will have the same shape as those shown for  $-\overline{uv}^+$ . Therefore, the observations drawn for  $-\overline{uv}^+$  are also valid for  $\overline{\theta v}^+$ .

The distributions of  $k^+$  and  $-\overline{uv}^+$  across the boundary layer are compared in Figs. 7 and 8, respectively. The results of the  $M_\infty = 10.31$  case with adiabatic wall boundary condition are not shown because  $k$ - $\omega$  model prediction of this case has not been attempted due to the high free-stream Mach number. In all cases shown,  $k$ - $\omega$  model over-predicts  $k^+$  and  $-\overline{uv}^+$  in the outer part of the boundary layer compared to the calculations of  $k$ - $\epsilon$  model/1. The over-prediction extends across the range,  $0.2 \leq y/\delta \leq 1.0$ . This is further evidence that the  $k$ - $\omega$  model is not a suitable model for compressible flat plate boundary-layer flows. Reduction of turbulence activities in the outer part of the boundary layer is clearly evident when either compressibility or wall cooling effects are present. The reduction increases as  $M_\infty$  increases. Therefore, it is expected that turbulence activities will be substantially reduced in a flow where the free-stream Mach number is large and the wall is highly cooled.

The near-wall  $k$  budget for case 73050504 is plotted in Fig. 9. Other budget plots are not shown because they are essentially similar to that given in Fig. 9. It can be seen that the  $k$  budget bears a lot of similarity with that calculated for incompressible flows (see e.g. Ref. 27). The additional compressible terms have negligible effect on the near-wall  $k$  budget. Therefore, the assumptions made to derive the near-wall function  $\xi$  in the dissipation rate equation are justified. Again, viscous diffusion balances dissipation at the wall. This balance extends to about  $y_w^+ = 4$  where turbulent diffusion and production become important. In the region,  $4 \leq y_w^+ \leq 15$ , viscous and turbulent diffusions, production and dissipation are equally important. Beyond  $y_w^+ = 15$ , production and dissipation are in balance, just as in the case of incompressible flows. Consequently, the near-wall turbulence behavior is very similar for both incompressible and compressible flows.



According to Refs. 25 and 27, Taylor series expansions about  $y_w^+ = 0$  can be assumed for  $k^+$ ,  $-\overline{uv}^+$  and  $-\overline{\theta v}^+$ . For incompressible flows, the expansions are valid up to about  $y_w^+ = 7$ . This range may not be applicable for compressible flows. Nevertheless, such expansions for small  $y_w^+$  can still be assumed. With the help of (21) these expansions can be written as:

$$k^+ = a_k(y_w^+)^2 + b_k(y_w^+)^3 + \dots, \quad (40a)$$

$$-\overline{uv}^+ = a_{uv}(y_w^+)^3 + b_{uv}(y_w^+)^4 + \dots, \quad (40b)$$

$$-\overline{\theta v}^+ = a_{v\theta}(y_w^+)^3 + b_{v\theta}(y_w^+)^4 + \dots, \quad (40c)$$

where the a's and b's are time-average coefficients that are functions of  $x$ . A similar expansion can be deduced for  $\epsilon^+$ . Again, using (21), the definition of  $\epsilon$  and its wall boundary condition, the expansion for  $\epsilon^+$  can be written as:

$$\epsilon^+ = 2a_k + 4b_k y_w^+ + \dots \quad (41)$$

From these expansions, it can be easily deduced that  $k^{+2}/\epsilon^+(y_w^+)^2 = 0.5$ . Therefore, the asymptotic behavior of  $k^{+2}/\epsilon^+(y_w^+)^2$  is 0.5 and is independent of  $M_\infty$  and wall thermal boundary conditions. The accuracy in which a model can predict this quantity is a reflection of the asymptotic consistency of the model. Table 1 shows that k- $\epsilon$  model/1 is indeed asymptotically consistent while k- $\epsilon$  model/2 is not as good.

The "a" coefficients can be determined from the calculations and their values are also listed in Table 1 for comparison. It can be seen that  $a_k$  varies with free-stream Mach number. A plot of  $a_k$  versus  $M_\infty$  for the adiabatic wall boundary condition is shown in Fig. 10. The value of  $a_k$  for the incompressible case is taken from Ref. 27 and is plotted at  $M_\infty = 0$ . Clearly, the trend is to approach an asymptotic value for  $a_k$  at high  $M_\infty$ . This decrease in  $a_k$  is a cause for the decrease in  $k$  for high Mach number flows (Figs 3 and 7). The physical reason is that compressibility tends to hinder turbulence mixing. As a result, both turbulent shear stress and kinetic energy decrease

significantly as  $M_\infty$  increases (Figs. 3, 5, 7 and 8). Since  $\epsilon_w^+ = 2a_k$  according to (41), dissipation at the wall is also dependent on  $M_\infty$ . There is no clear trend for  $a_{uv}$  and  $a_{v0}$ . However, the values of  $a_{uv}$  are consistent with that calculated for incompressible flows<sup>27</sup> and direct simulation data<sup>35</sup>. The value of  $a_{v0}$  is essentially zero. Since there is no data available for verification, its correctness is in doubt.

The inadequacy of the  $k-\omega$  model is further investigated by comparing its calculations with cases where the free-stream Mach number is about 5 or less. These calculations are carried out to evaluate the applicability of the  $k-\omega$  model at low  $R_\theta$ . The  $R_\theta$  for cases 53011302 and 59020105 are 5532 and 3936, respectively. Case 53011302 has adiabatic wall boundary condition and  $M_\infty = 4.544$  while case 59020105 has constant wall temperature and  $M_\infty = 5.29$ . These Reynolds number may be too low for the  $k-\omega$  model because it is not specifically formulated for low Reynolds-number flows. In order to substantiate this point, the  $k-\omega$  model is used to calculate case 53011302 up to an  $R_\theta = 14,950$  and case 59020105 up to an  $R_\theta = 14,000$ . These results and the calculations carried out to the measured  $R_\theta$ 's are plotted in Figs. 11 and 12 for comparison. The log-law is also shown in Fig. 11 to illustrate the calculated log behavior of the compressible boundary layers. It is evident that the  $k-\omega$  model performs better at high  $R_\theta$  and the calculated log behavior moves closer to that of the log-law (Fig. 11). The discrepancy between model calculations and measurements is decreased substantially when  $R_\theta$  is increased by a factor of about three. For the cooled wall case, even though there is improvement, it is not enough to bring the calculations to a closer agreement with measurements. There is still significant difference between this calculation and that due to  $k-\epsilon$  model/1 (compare Fig. 1d with Fig. 12a). However, there is little or no improvement in the prediction of mean temperature (compare Fig. 2d with Fig. 12b).

Finally, the ability of the  $k-\epsilon$  models to predict skin friction coefficient over a range of  $M_\infty$  and wall temperatures is illustrated in Fig. 13 and Table 2. In Fig. 13, the variation of  $C_f/(C_f)_i$  with  $M_\infty$  for the case of adiabatic wall boundary condition is shown. Here,  $(C_f)_i$  is the skin friction coefficient for an incompressible flow evaluated at the same  $R_\theta$  and is determined to be

$2.73 \times 10^{-3}$ . The calculations shown are from k- $\epsilon$  model/1. Below  $M_\infty = 5$ , the calculated k- $\epsilon$  model/2 variation of  $C_f/(C_f)_i$  with  $M_\infty$  is essentially the same as that shown in the figure. There is a slight decrease of 1 - 2% for  $M_\infty$  beyond 5. This means that both k- $\epsilon$  model/1 and k- $\epsilon$  model/2 predict the same  $C_f/(C_f)_i$  variation with  $M_\infty$  for adiabatic wall boundary condition. The predictions for the cooled wall case are not as good (Table 2). Here, substantial error starts to accumulate at a temperature ratio of about  $\Theta_w/\Theta_r = 0.6$ . The error at this point is about 3% but it quickly grows to about 9% at  $\Theta_w/\Theta_r = 0.4$  and becomes even larger as  $\Theta_w/\Theta_r$  decreases. This trend is contrary to previous calculations<sup>13</sup>. The disagreement could be attributed to the assumption of a constant turbulent Prandtl number. If  $Pr_t = 0.7$  is assumed, the calculations are in better agreement with data up to  $\Theta_w/\Theta_r = 0.4$ , but beyond this point significant error in prediction still exists. The reason could be a reduction of turbulent mixing as a result of a cooled wall. However, this effect has not been appropriately accounted for in the models, particularly their near-wall behavior. In other words, if highly cooled-wall flows are to be predicted correctly, heat fluxes should be modeled separately rather than linking to momentum fluxes via a constant turbulent Prandtl number.

### 6.3 Discussion

The following observations can be drawn from the above analysis. Firstly, Morkovin's hypothesis is valid up to a free-stream Mach number of about 5 for flat plate boundary-layer flows with adiabatic wall boundary condition. This means that the effects of fluctuating density are becoming more and more important as  $M_\infty$  increases beyond 5. Secondly, the assumption of a constant turbulent Prandtl number is not appropriate for cooled wall thermal boundary condition. The reason is the reduction in mixing due to a cooled wall and this effect is not correctly accounted for in a constant turbulent Prandtl number approach. Most likely a heat flux model is required if the characteristics of cooled-wall compressible boundary-layer flows are to be predicted correctly. Thirdly, it is important to model the near-wall flow correctly if the overall boundary-layer characteristics are to be predicted with confidence. The failings of the k- $\omega$  model can be attributed to the asymptotically inconsistent near-wall corrections implemented into the governing equations.

This point is substantiated by the present  $k$ - $\epsilon$  model calculations where all additional compressible terms in the turbulence equations are neglected. The results are in good agreement with measurements even though they differ slightly from the predictions of the model calculations where all the additional terms are retained. In other words, an asymptotically consistent near-wall model is more important to the prediction of compressible boundary-layer flows than the inclusion of fluctuating density effects in the modeled equations. Finally, the predicted near-wall characteristics are very similar to those calculated for incompressible flows. In the range of free-stream Mach number tested, the calculated near-wall characteristics are essentially independent of Mach number and wall thermal boundary condition. Very near the wall, viscous diffusion of  $k$  is balanced by the dissipation of  $k$ . Beyond  $y_w^+ = 15$ , dissipation is balanced by mean shear production of  $k$ . In between these two regions, viscous and turbulent diffusion of  $k$ , production of  $k$  and dissipation of  $k$  are of importance in the budget of  $k$ . The additional compressible terms in the  $k$ -equation are essentially negligible in the near-wall region up to  $y_w^+ = 50$ . Perhaps, this is the reason why the model also performs well when the additional compressible terms are neglected in the equations.

## 7. Conclusions

The  $k$  and  $\epsilon$  equations for compressible flows are derived by assuming that there is no dynamical similarity between the compressible and incompressible fields. Therefore, the influences of fluctuating density on the mean and turbulence fields have to be accounted for in the modeled equations. This can be accomplished by first re-casting the exact  $k$  and  $\epsilon$  equations into forms that are similar to their incompressible counterparts. In other words, the viscous diffusion and dissipation functions have to be defined exactly like their incompressible terms. This procedure gives rise to additional terms in the  $k$  and  $\epsilon$  equations. These terms depend explicitly on compressibility and vanish when the fluid density becomes constant. One extra term in the  $k$ -equation is related to fluid dilatation and can be interpreted as compressible dissipation. The others are production terms that depend on the gradients of the mean pressure and mean viscous shears. All additional terms are found to be relatively unimportant in the near-wall region, or  $0 \leq y_w^+ \leq 50$ . However, away from this region, they are quite significant compared to other terms in the equations. This realization, therefore, allows the near-wall incompressible models to be extended directly to compressible flows without modifications, while still maintaining the balance of the modeled equations as a wall is approached. Models are proposed for the additional terms in the  $k$  and  $\epsilon$  equations. The constants introduced by the new models are determined by calibrating the calculations against measurements in compressible flows.

The near-wall two-equation model is used to calculate compressible flat plate boundary-layer flows with different wall thermal boundary conditions and free-stream Mach numbers. Comparisons are made with various mean flow measurements and with calculations of the  $k$ - $\omega$  model. Good agreement is obtained between the present calculations and measurements. In particular, the log-law for compressible flows is recovered and the slope of the log-law is found to be fairly independent of free-stream Mach number for the range,  $0 \leq M_\infty \leq 10$ , tested. However, this is not the case for the  $k$ - $\omega$  model. This particular model is formulated by assuming that compressibility effects can be accounted for fully by the variable mean density alone. Therefore,

the effects of fluctuating density on turbulent mixing are essentially neglected. Furthermore, the model is not internally consistent as a wall is approached. Consequently, its prediction of the near-wall flow is in error and this, in turn, leads to an incorrect prediction of the flow away from the wall. The discrepancies can be attributed to a near-wall behavior that is not asymptotically correct and to the neglect of fluctuating density effects.

The importance of an asymptotically correct near-wall model is illustrated by using the present  $k$ - $\epsilon$  model minus all the additional compressible terms in the equations to calculate the same experimental flows. Again, good agreement is obtained between calculations and measurements. The only disagreements come in the predictions of the near-wall asymptotic behavior of the turbulence quantities. Even then, the discrepancy only amounts to a few percentage of that calculated by the  $k$ - $\epsilon$  model with all additional compressible terms included in the equations. The calculated near-wall turbulence statistics are internally very consistent and approach their asymptotic values correctly. Furthermore, they are found to be in good agreement with direct simulation data obtained for incompressible flat plate boundary layers. This implies that variable mean density has little or no effects in the near-wall region of boundary-layer flows with free-stream Mach number up to 10. Therefore, the dynamical field in a compressible flow behaves like an incompressible one in the near-wall region. Finally, the Morkovin hypothesis is found to be valid up to a free-stream Mach number of about 5.

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Table 1. Asymptotic near-wall behavior of the turbulence properties.

Case	$M_\infty$	$\Theta_w/\Theta_r$	Model	$a_k$	$a_{uv} \times 10^4$	$a_{v\theta} \times 10^7$	$k^+/\epsilon^+(y_w^+)^2$
55010504	2.244	1.0	k- $\epsilon$ model/1	0.0987	7.168	-0.462	0.50
53011302	4.544	1.0	k- $\epsilon$ model/1	0.0844	6.160	-2.243	0.50
73050504	10.31	1.0	k- $\epsilon$ model/1	0.0787	6.830	4.550	0.50
73050504	10.31	1.0	k- $\epsilon$ model/2	0.0812	6.950	-70.4	0.51
59020105	5.29	0.92	k- $\epsilon$ model/1	0.0795	6.230	-6.823	0.50
59020105	5.29	0.92	k- $\epsilon$ model/2	0.0801	6.290	-13.17	0.50

Table 2. k- $\epsilon$  model/1 calculated skin friction coefficient for different wall temperature  
at  $M_\infty = 5.0$  and  $Re = 10^4$  with a  $(C_f)_i = 2.73 \times 10^{-3}$ .

$\Theta_w/\Theta_r$	Calculated $C_f \times 10^3$	Calculated $C_f/(C_f)_i$	$C_f/(C_f)_i$ from van Driest II
0.8	1.07	0.392	0.392
0.6	1.15	0.421	0.434
0.4	1.22	0.447	0.495
0.2	1.26	0.461	0.580

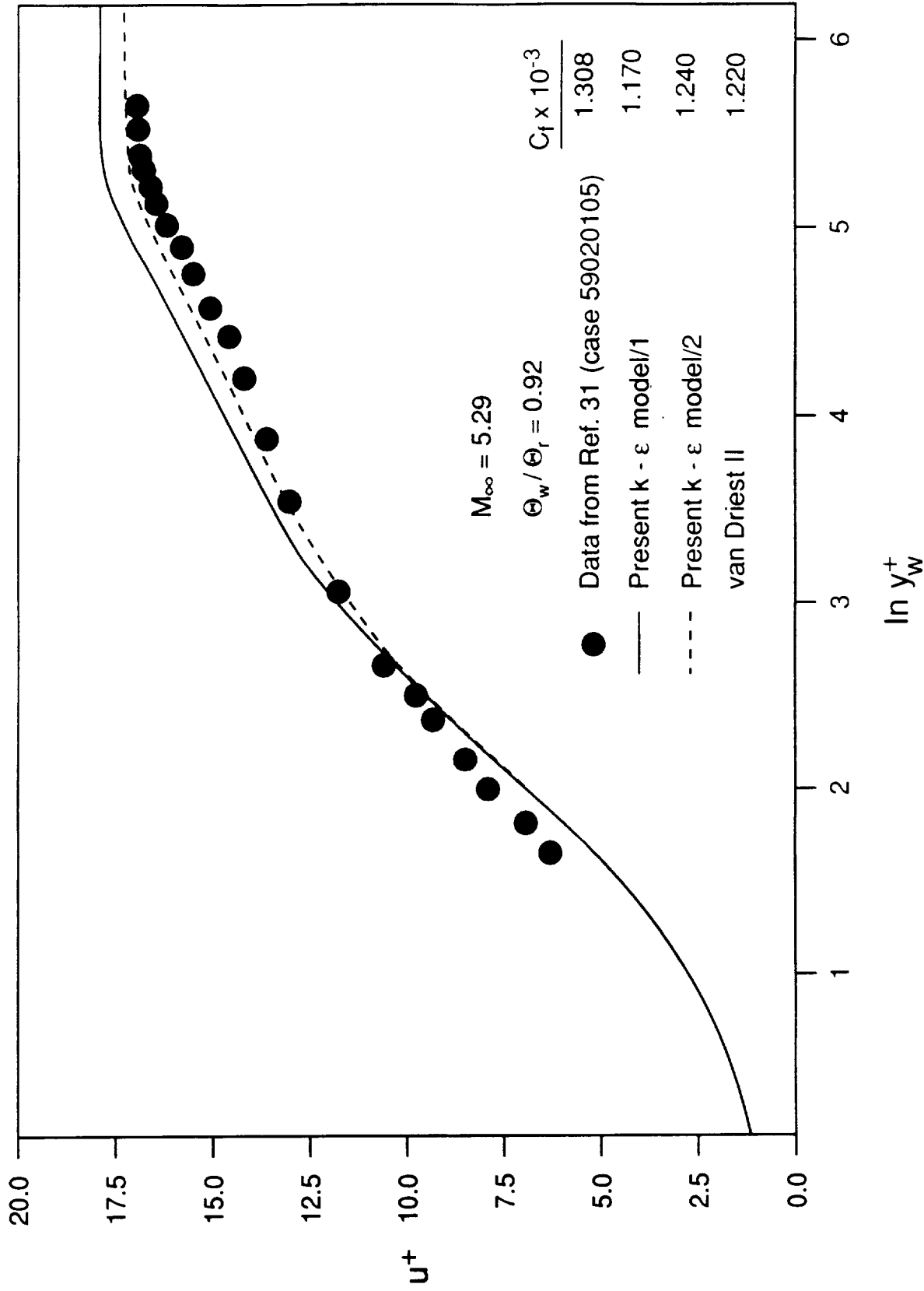


Figure 1a. Comparison of calculated mean velocities with measurements.

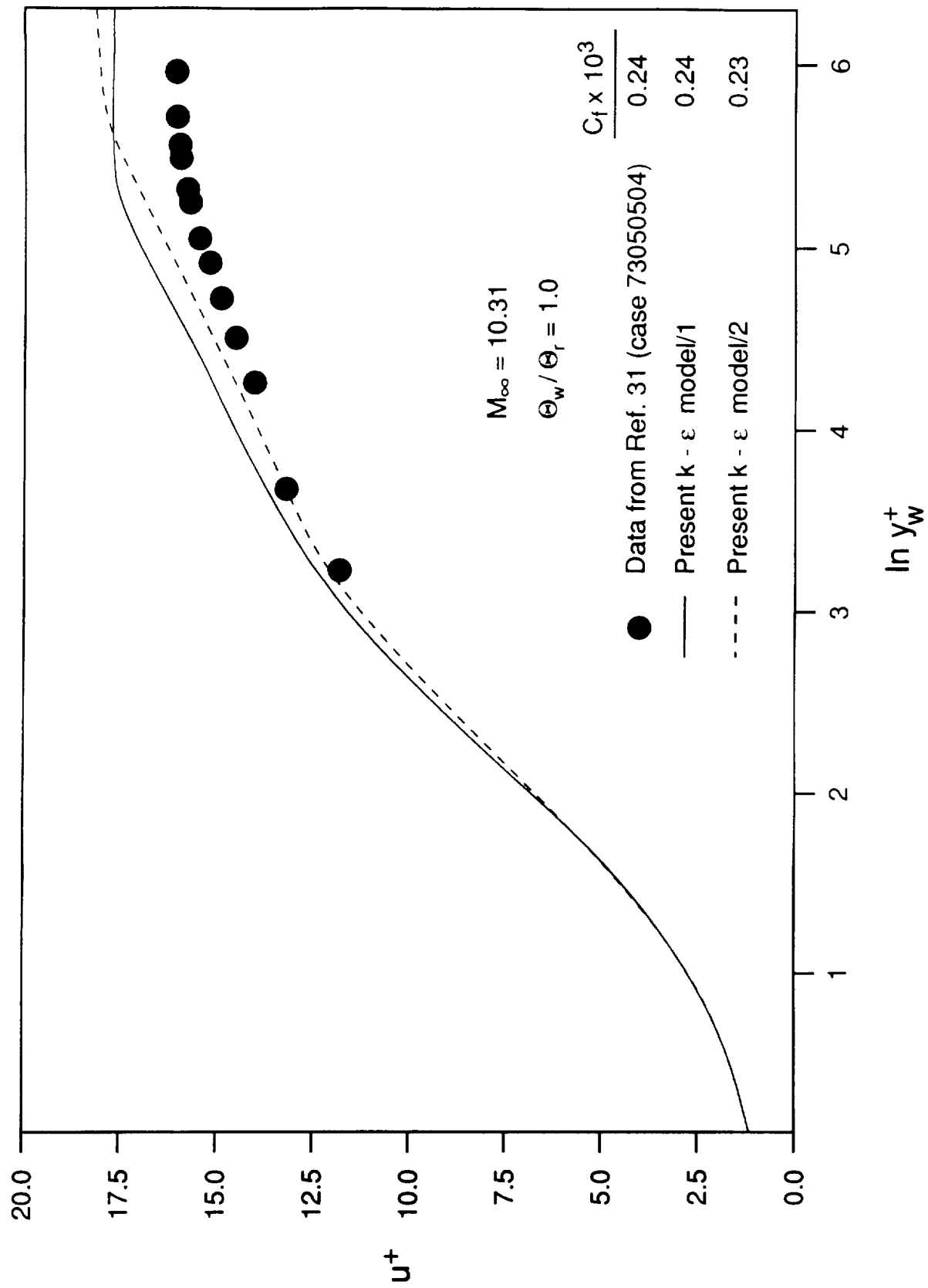


Figure 1b

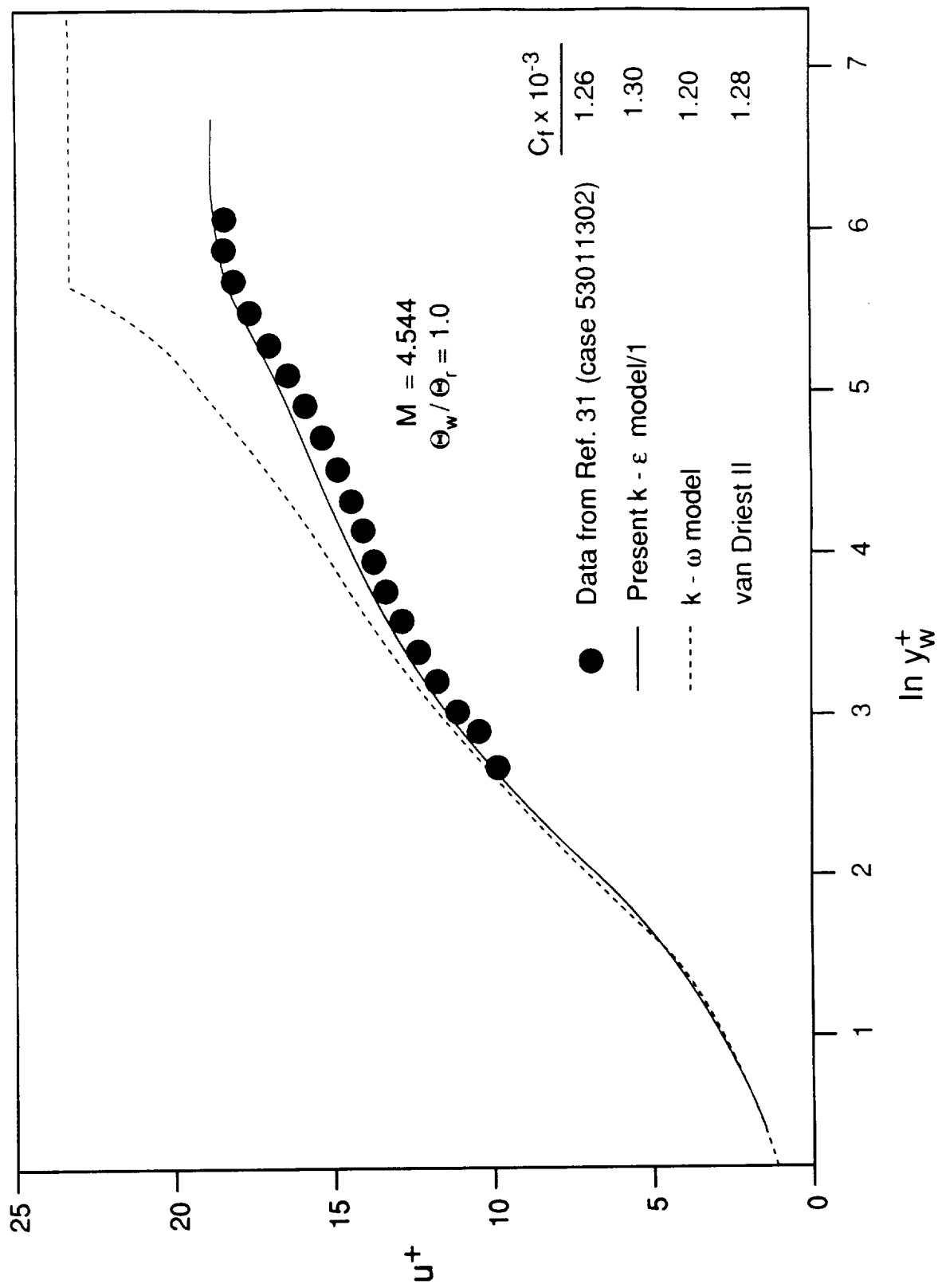


Figure 1c

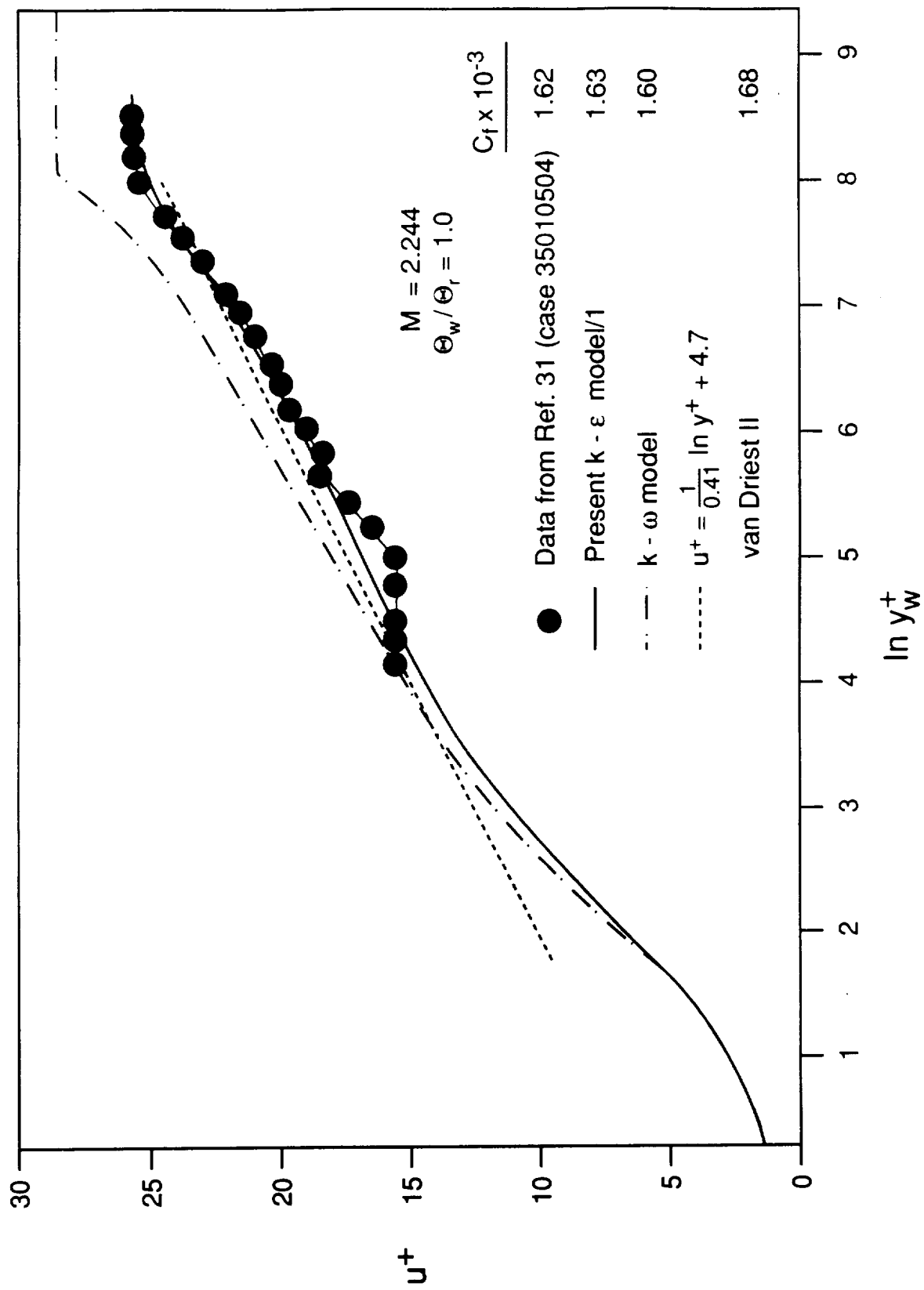


Figure 1d

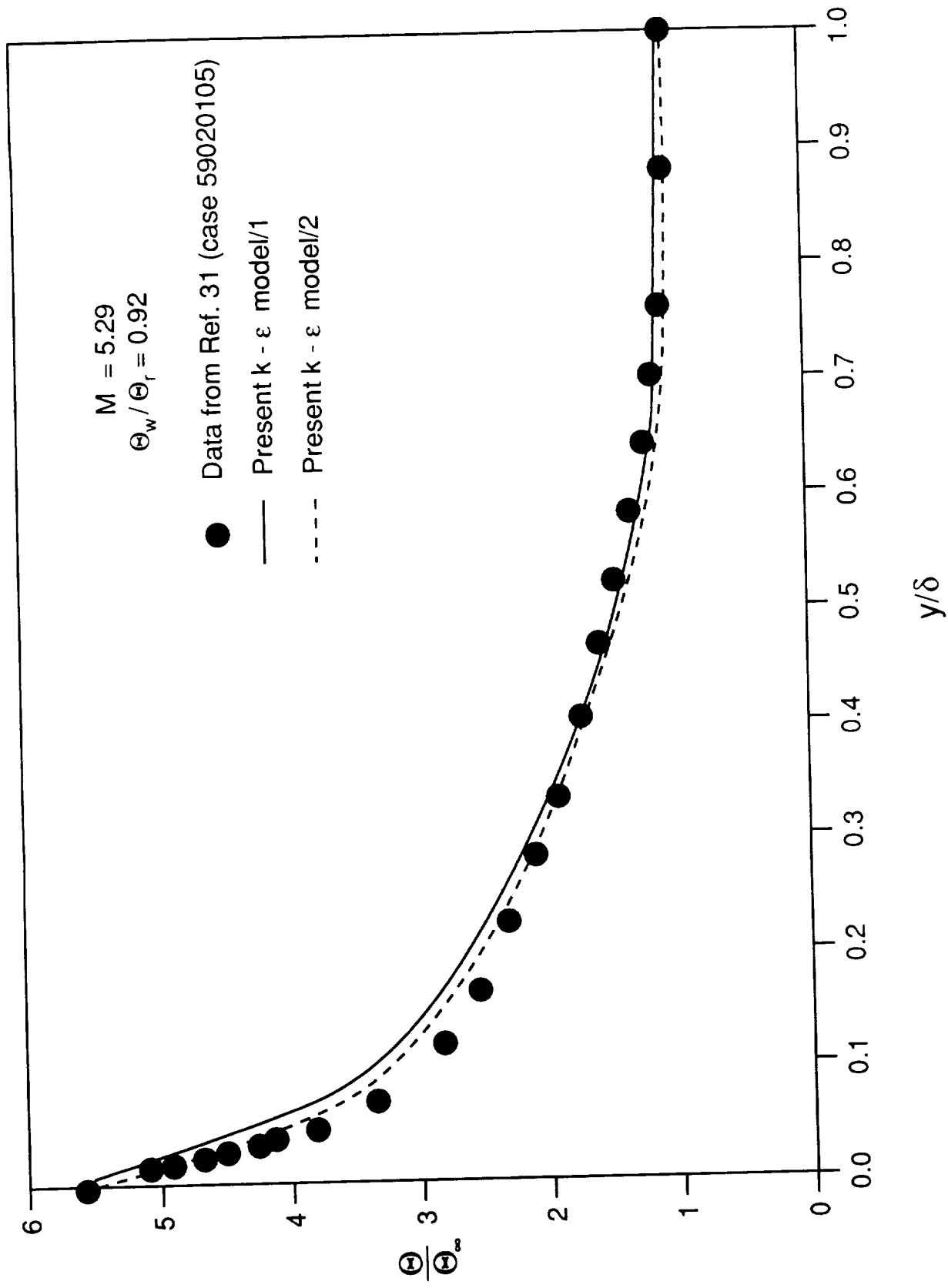


Figure 2a Comparison of calculated mean temperatures with measurements.



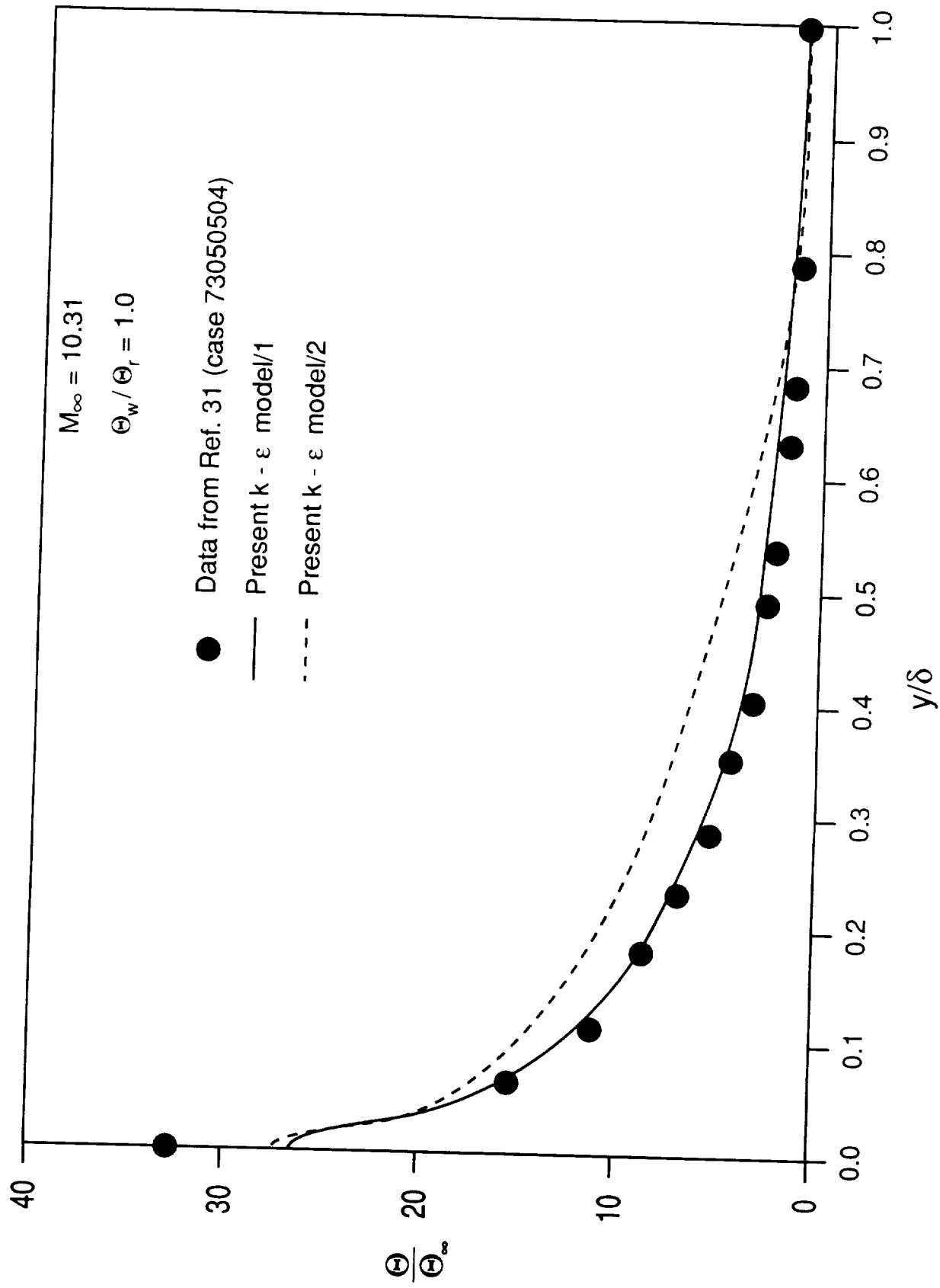


Figure 2b

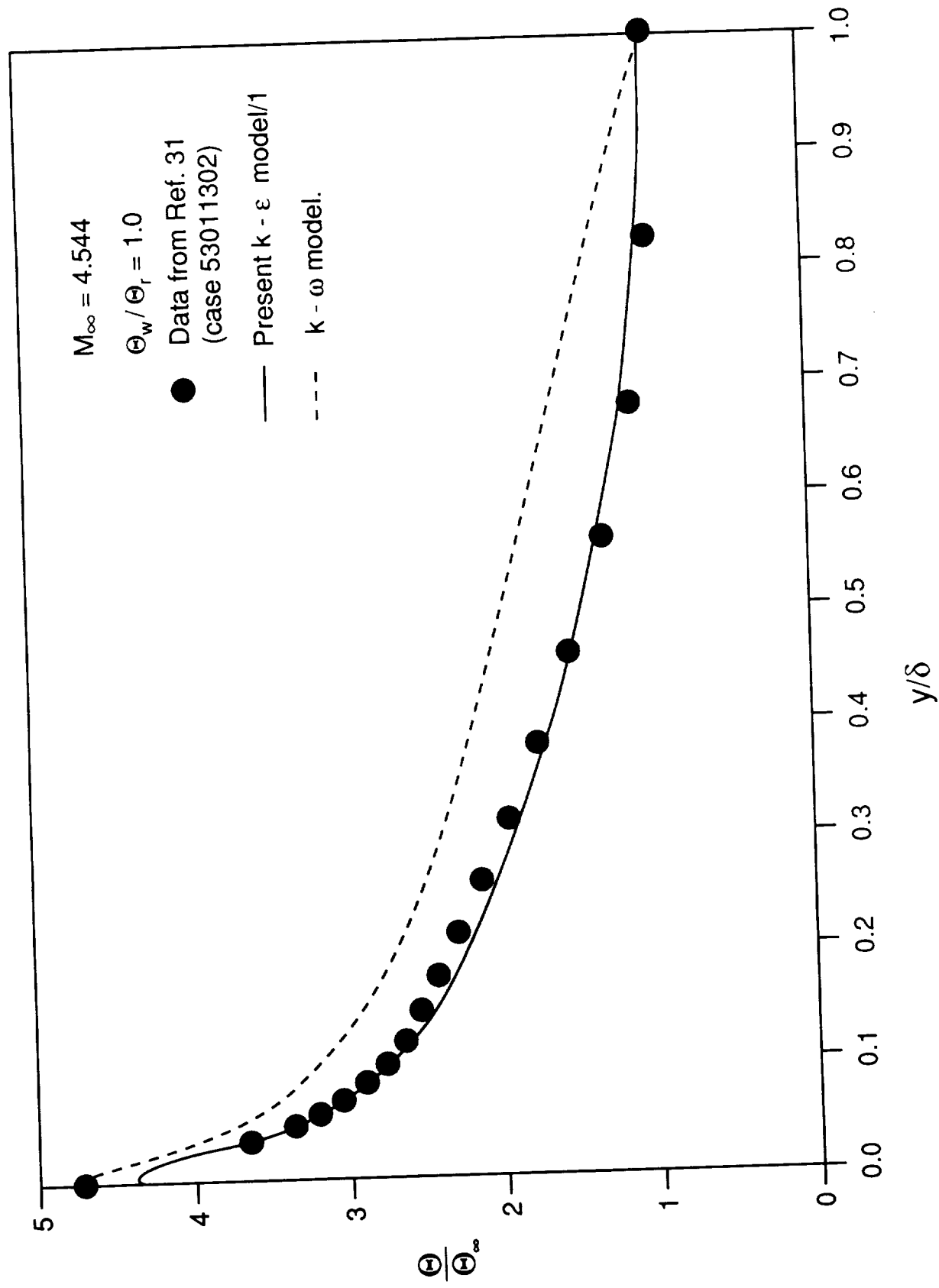


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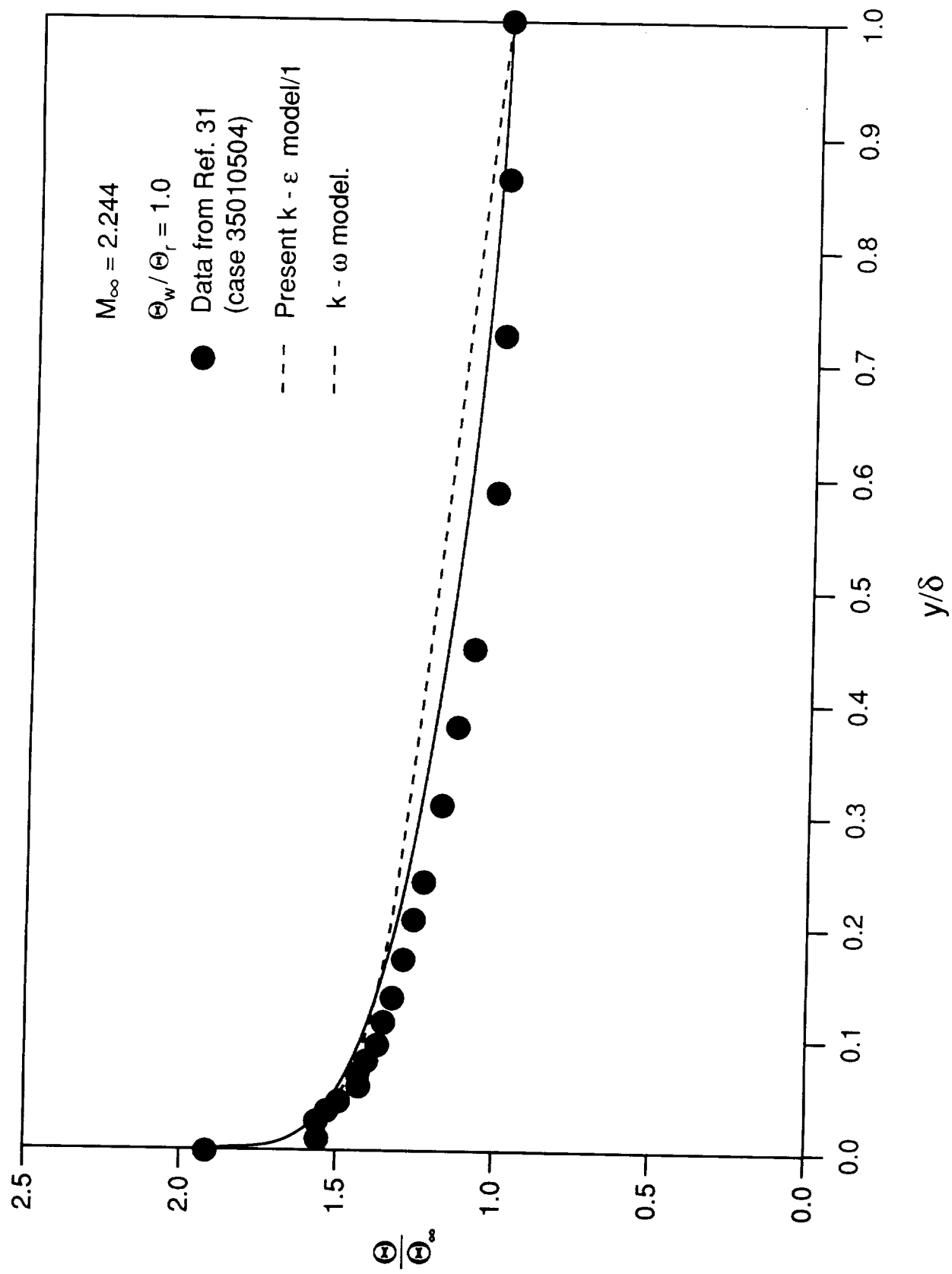


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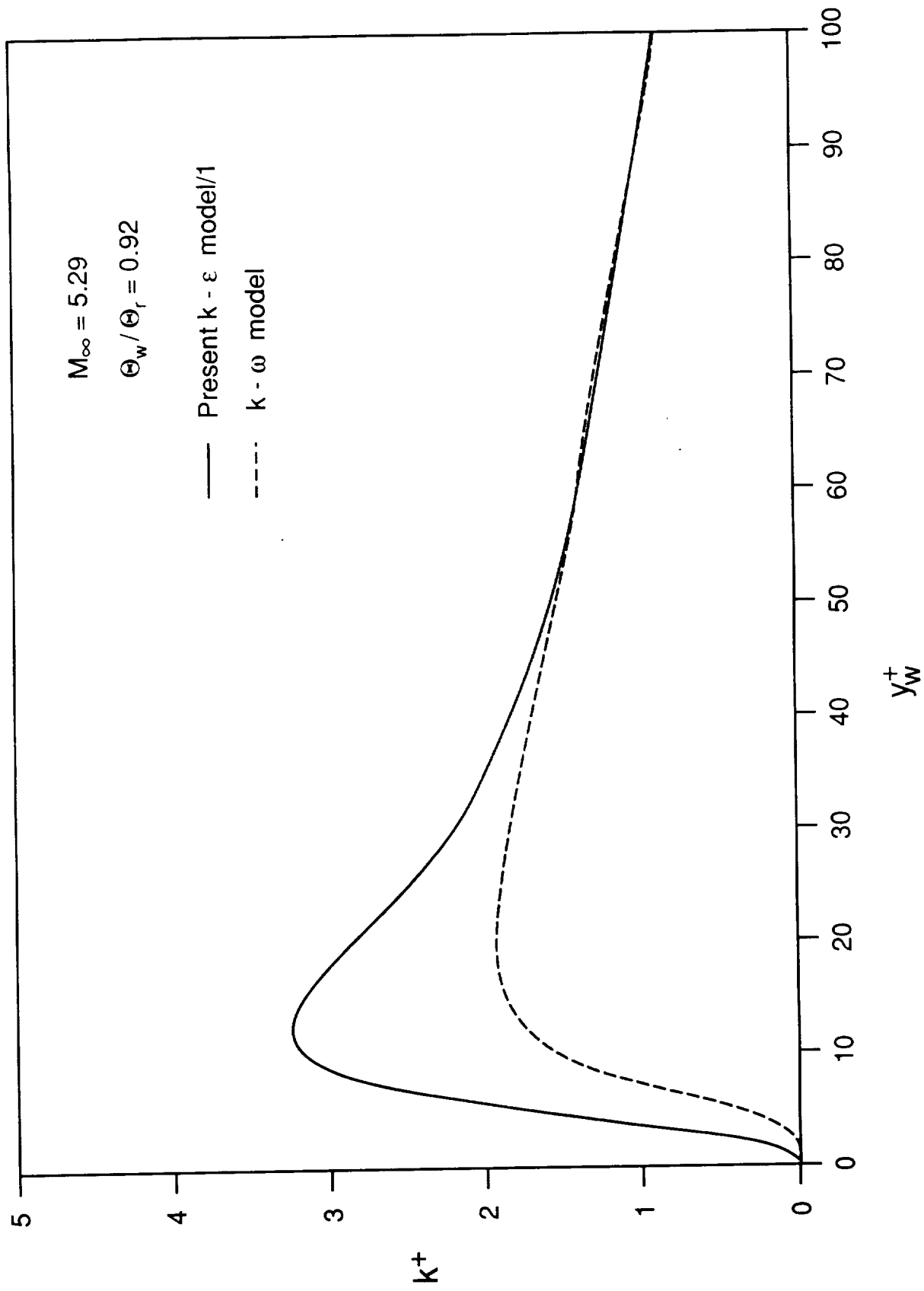


Figure 3a Near-wall distributions of turbulent kinetic energy.

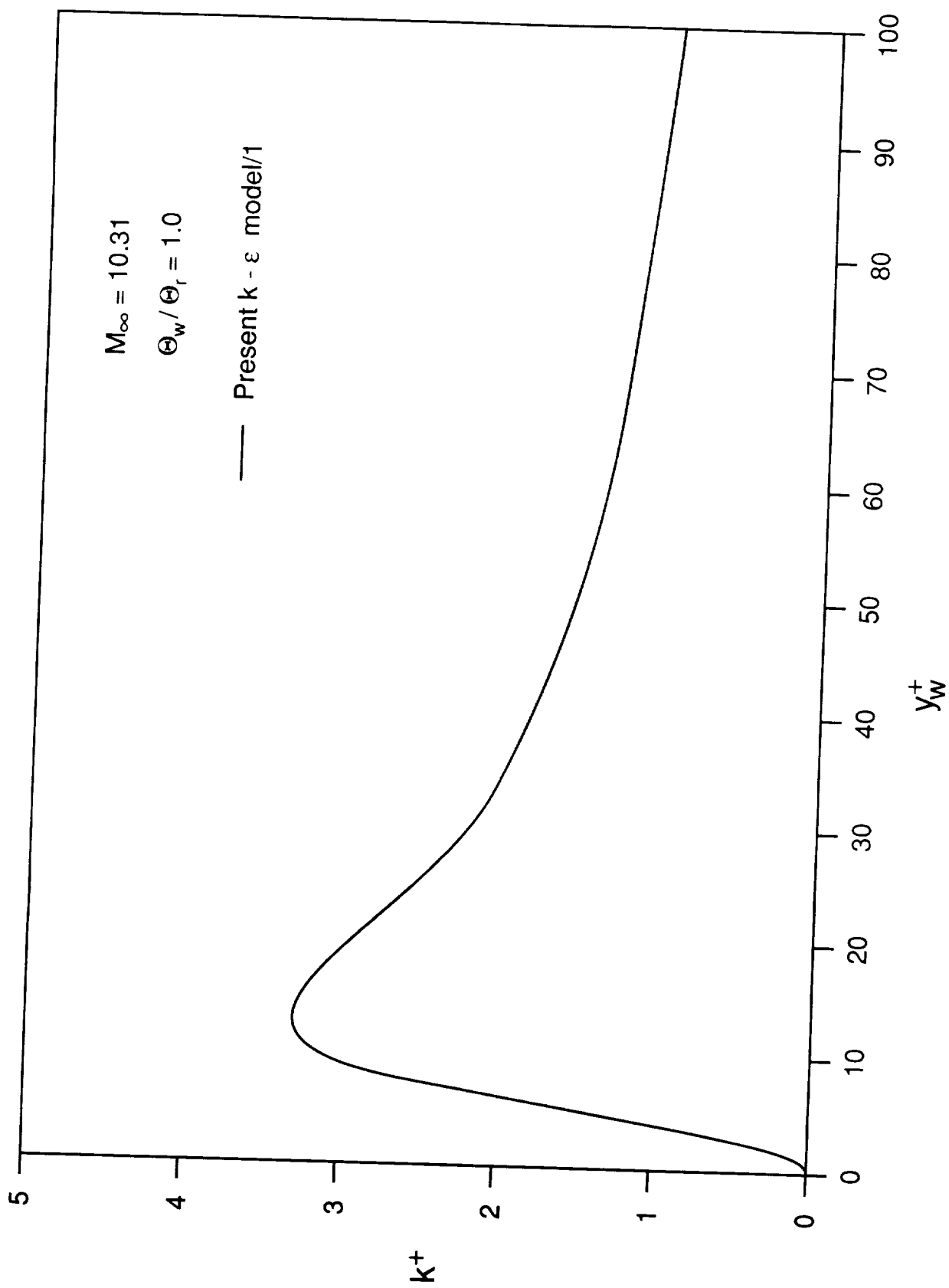


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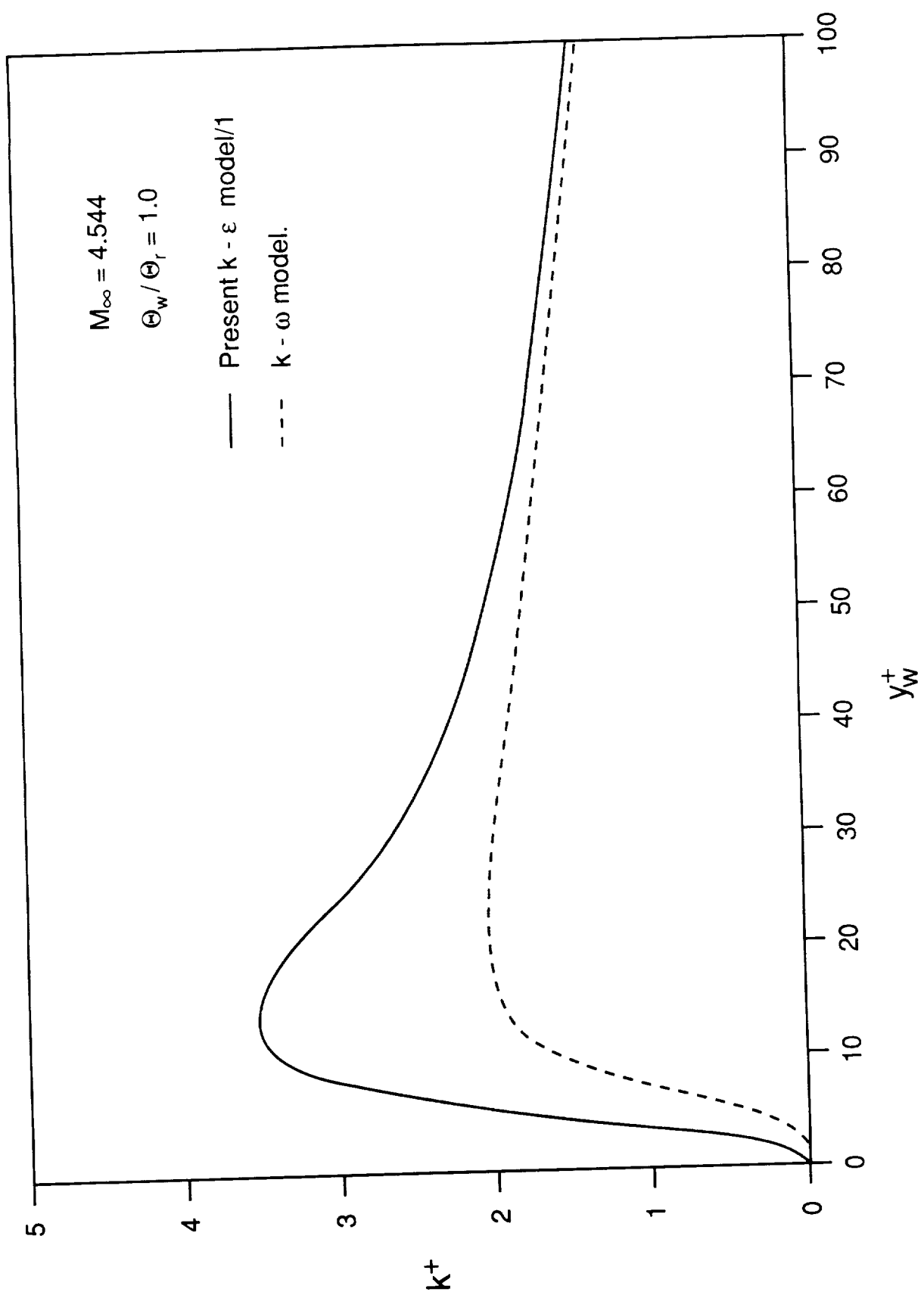


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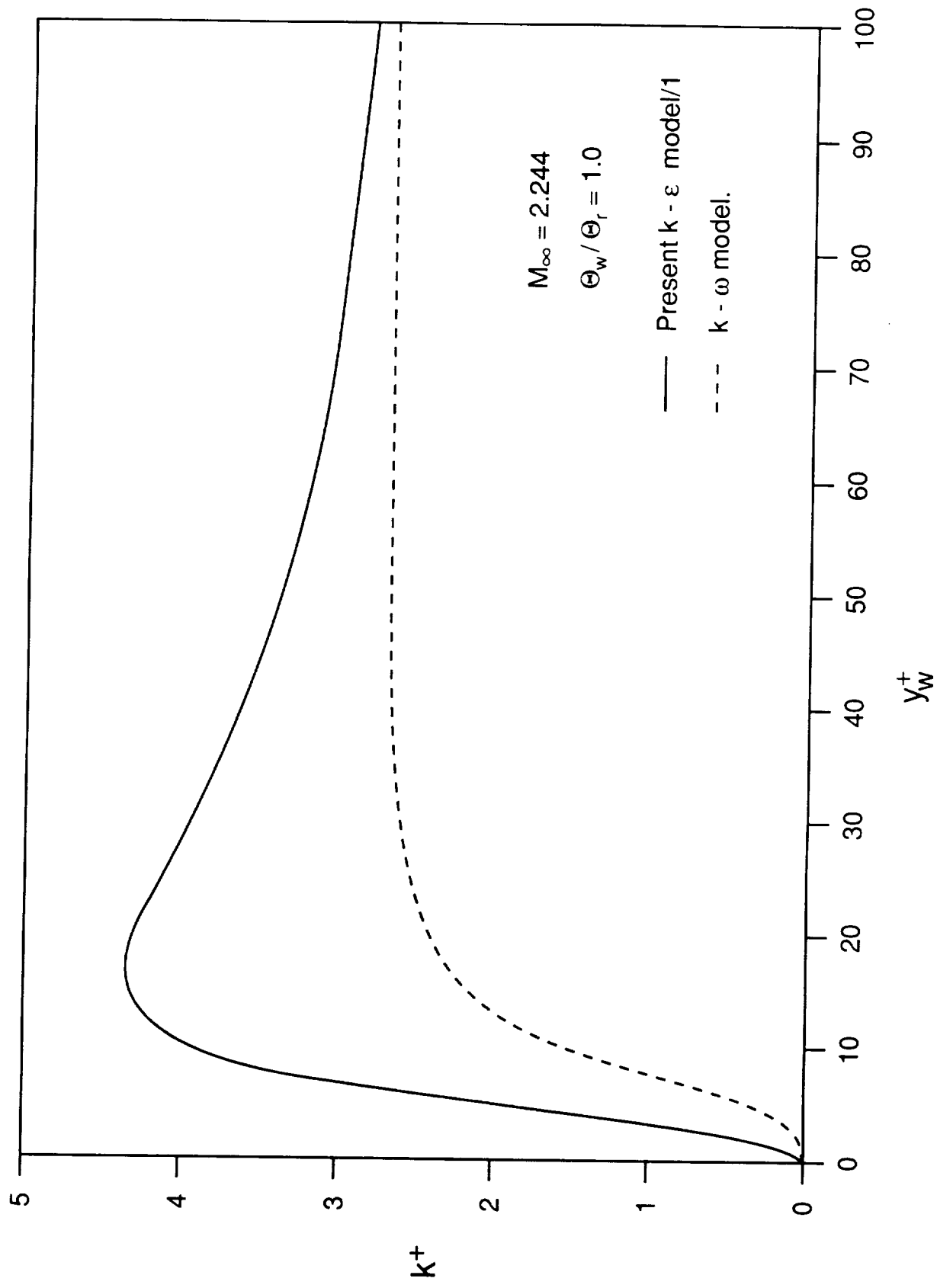


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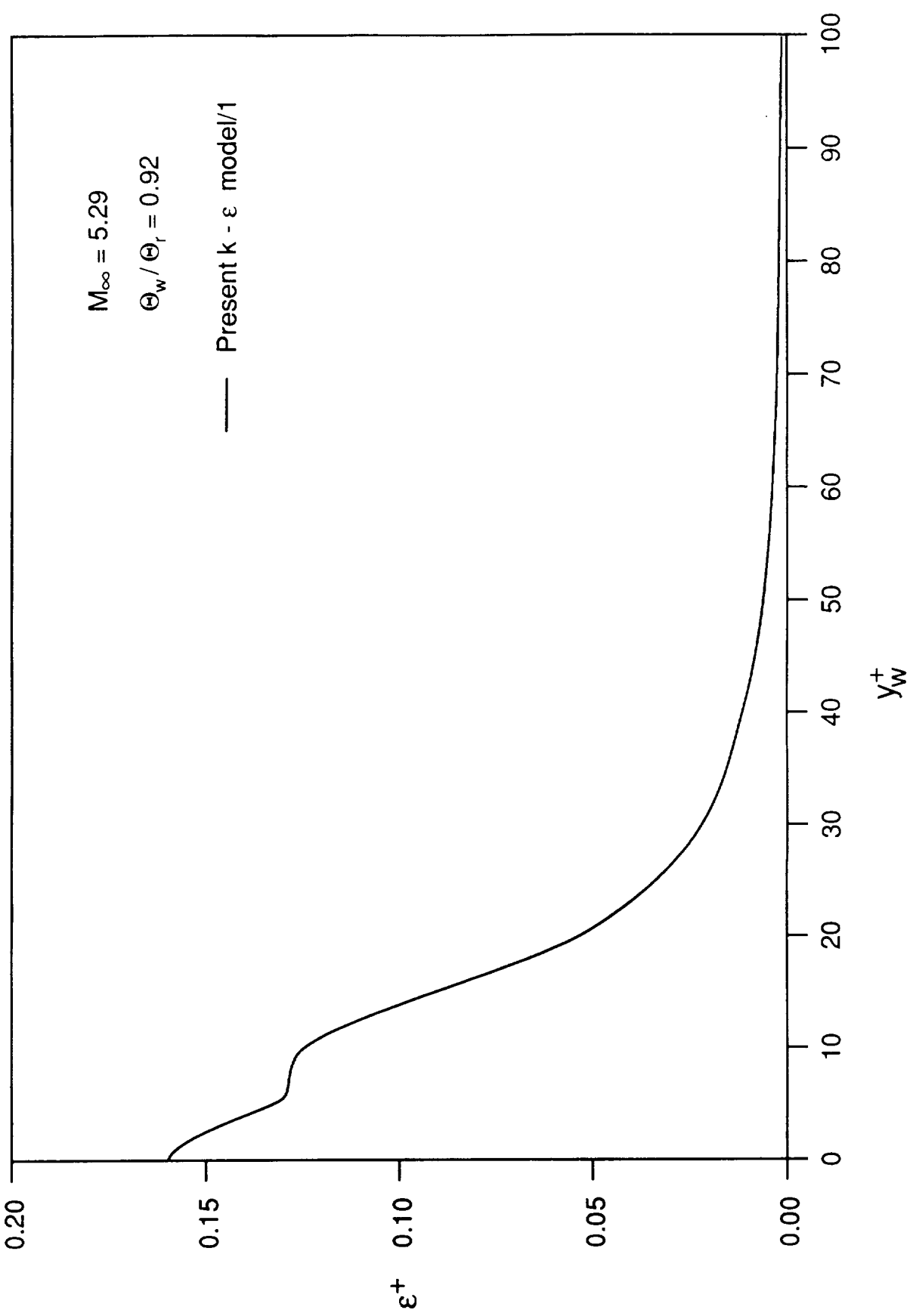


Figure 4a Near-wall distributions of the solenoidal dissipation rate of the turbulent kinetic energy.



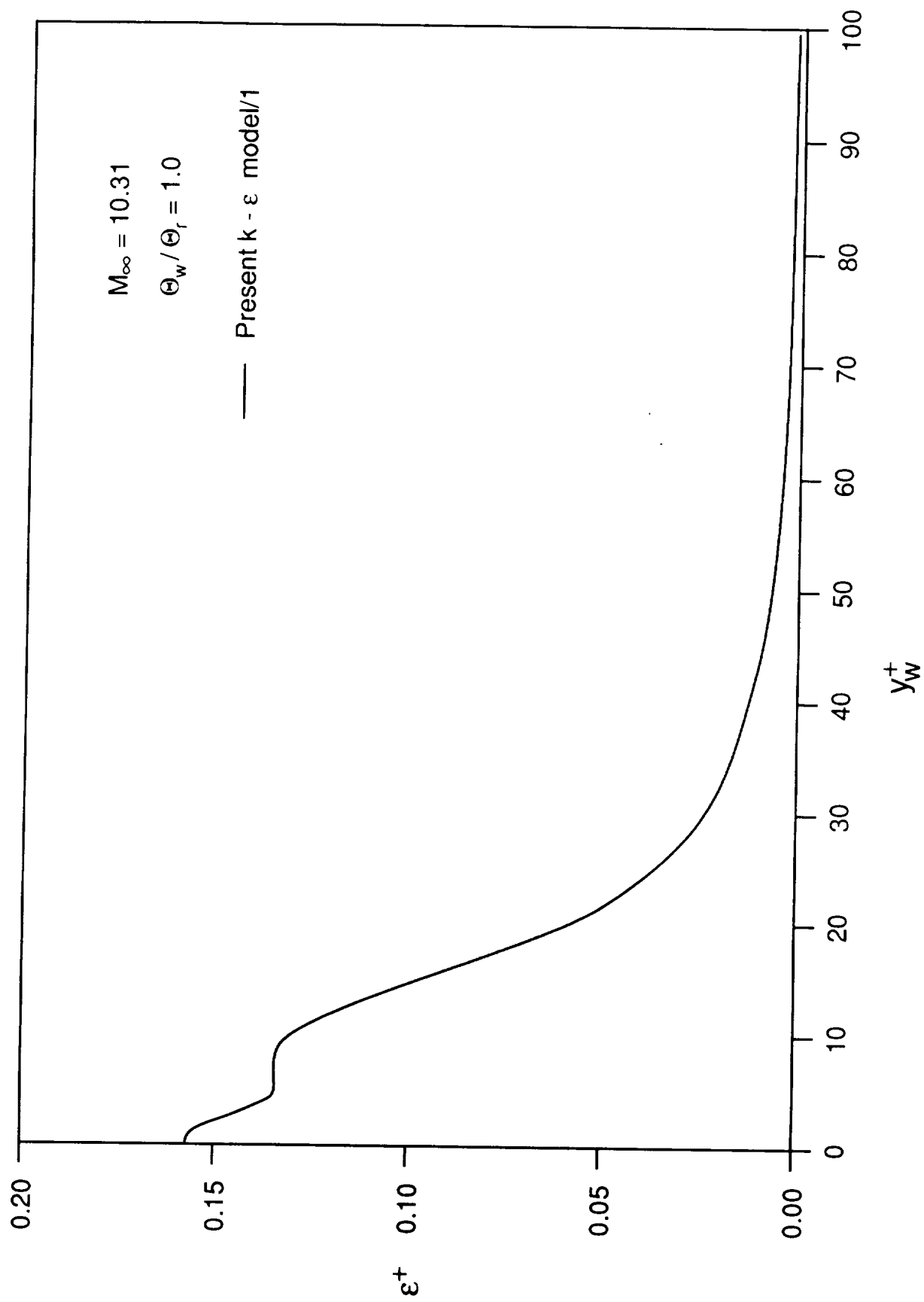


Figure 4b

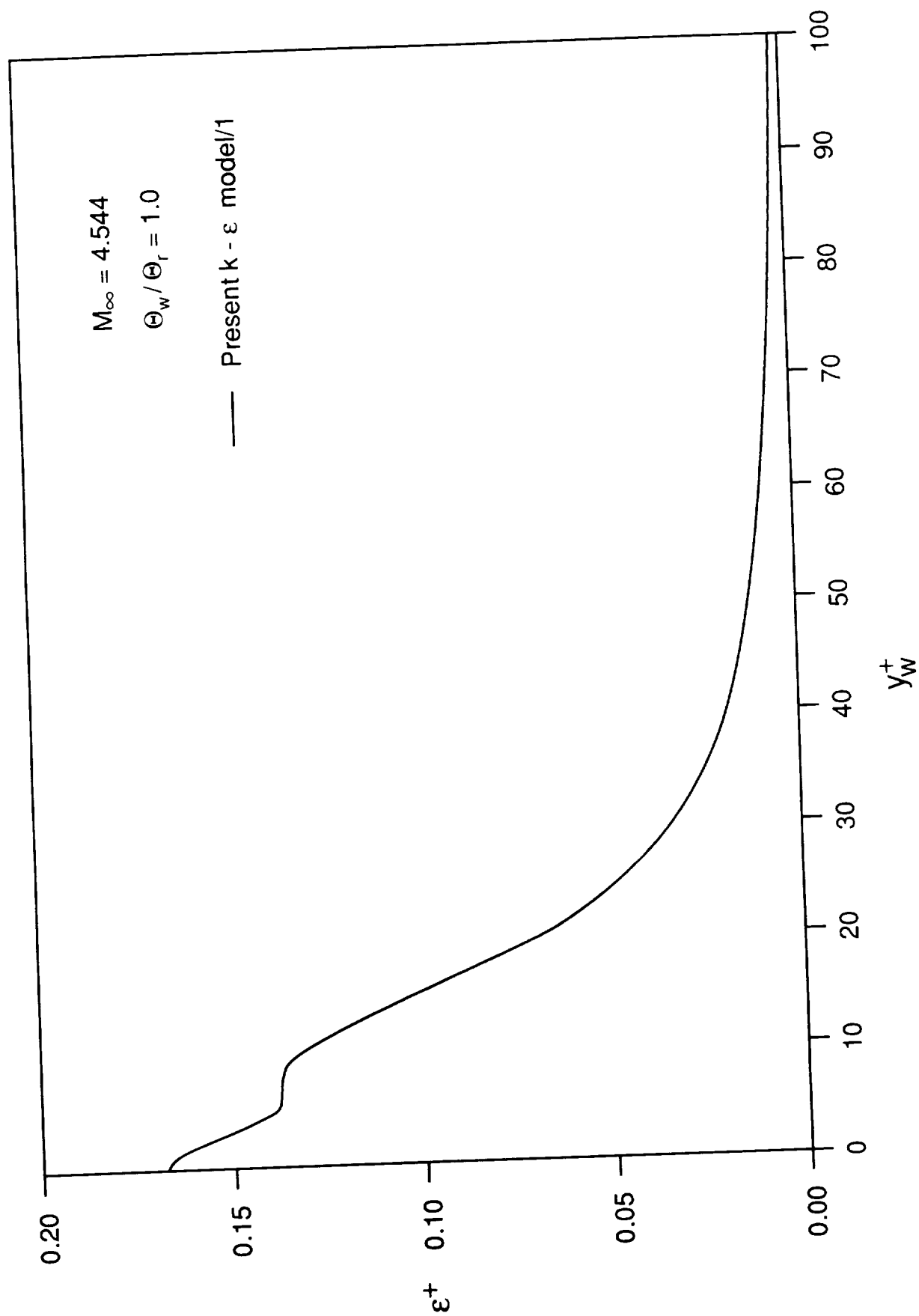


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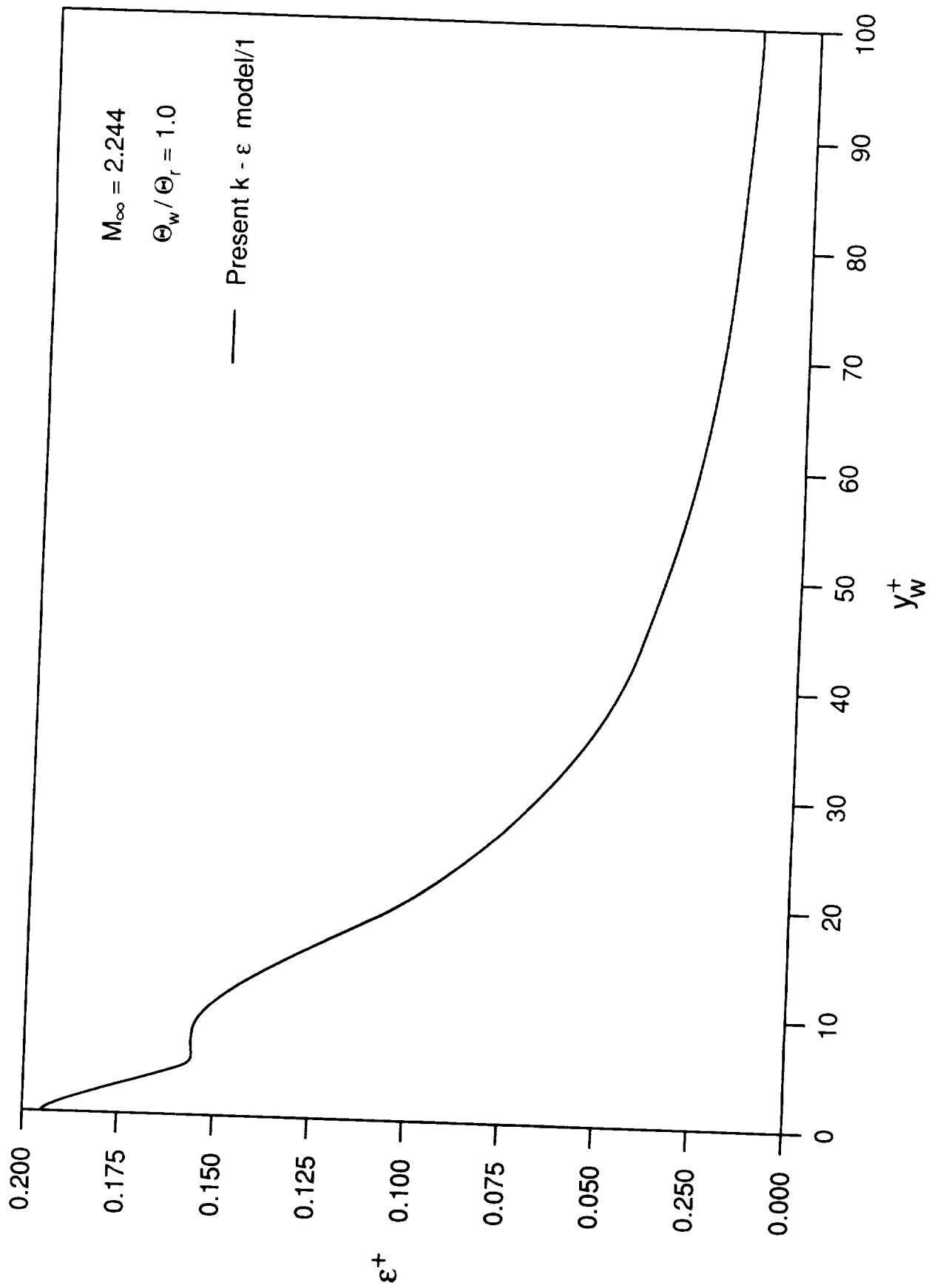


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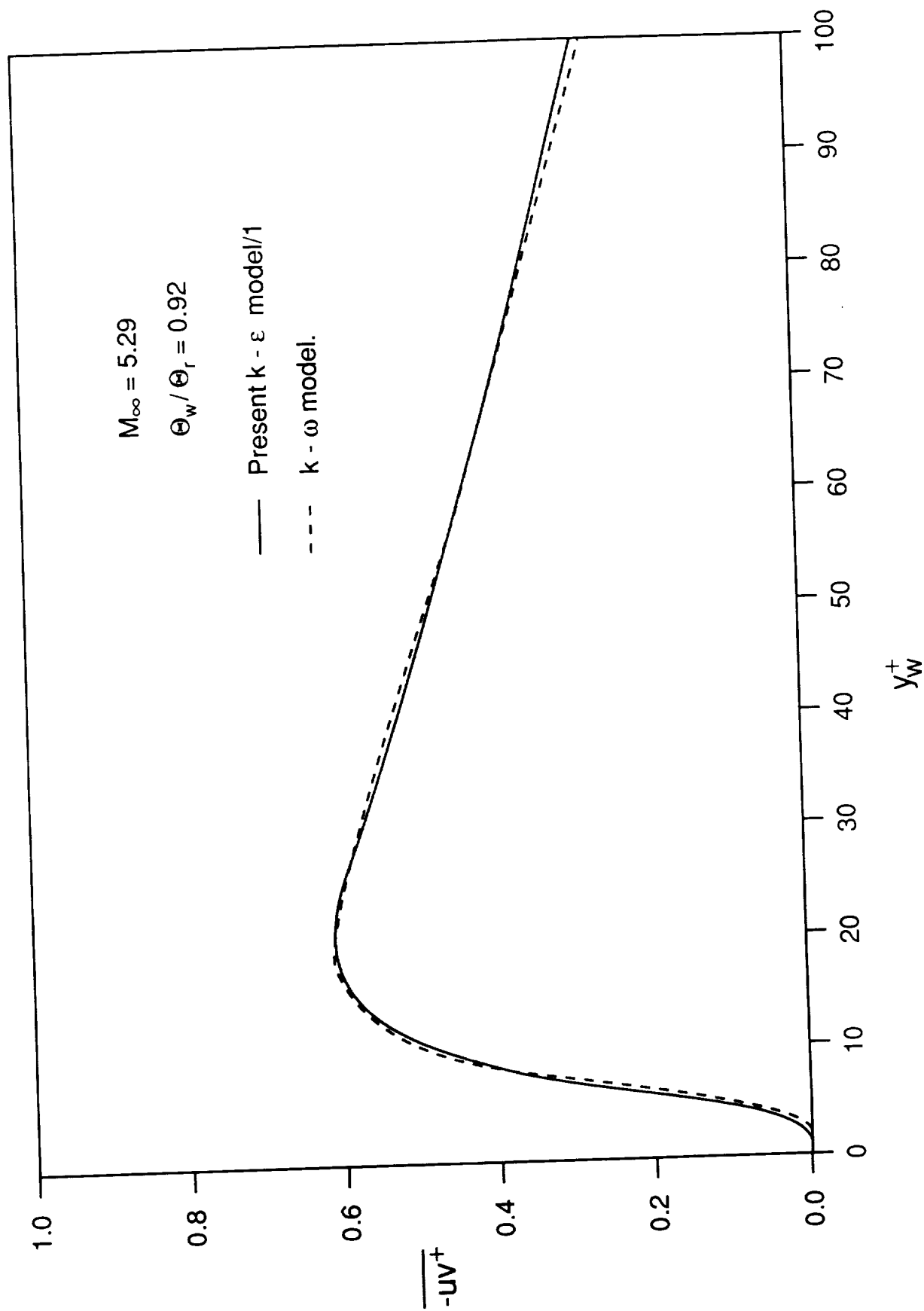


Figure 5a Near-wall distributions of Reynolds shear stress.

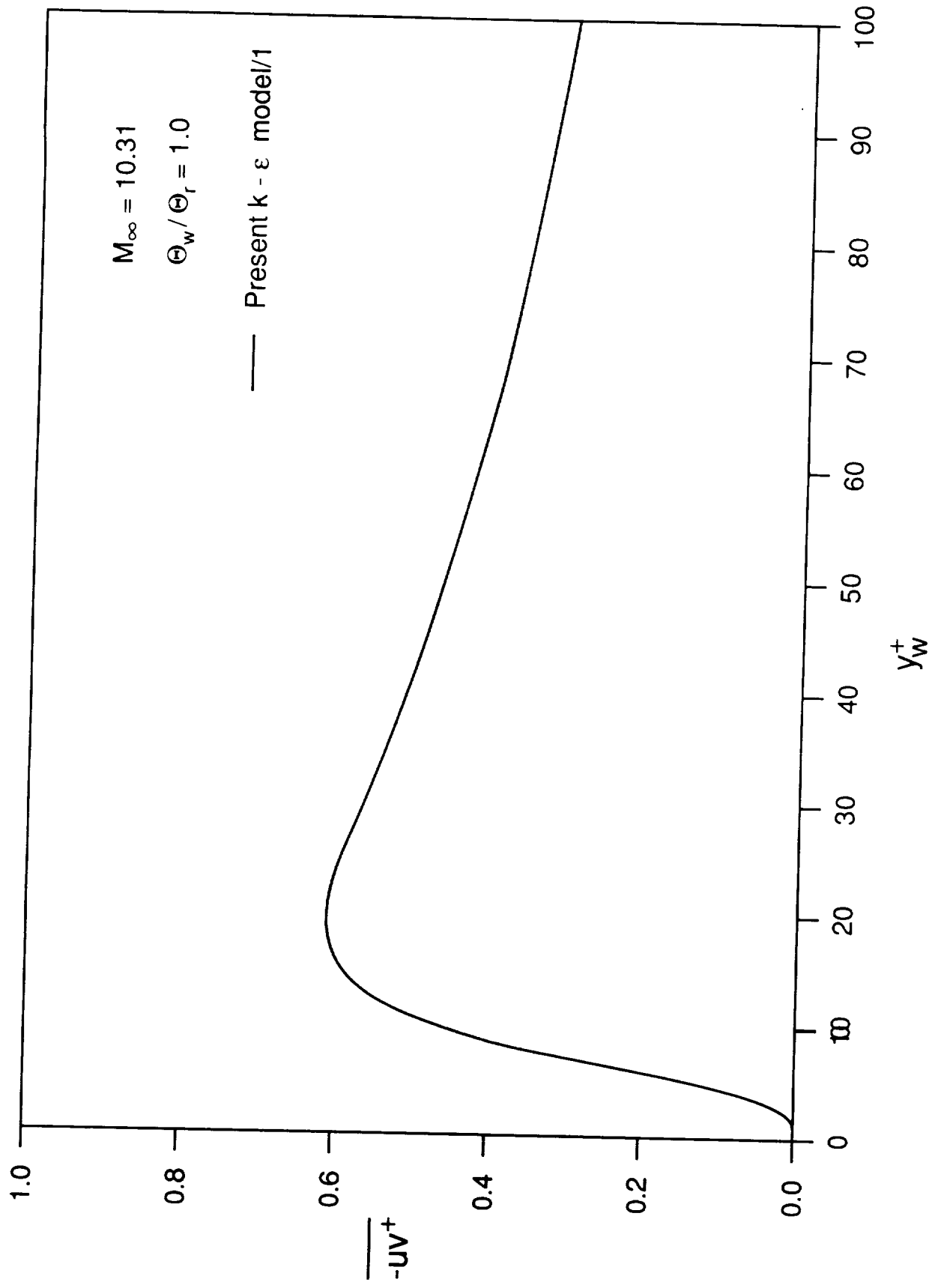


Figure 5b

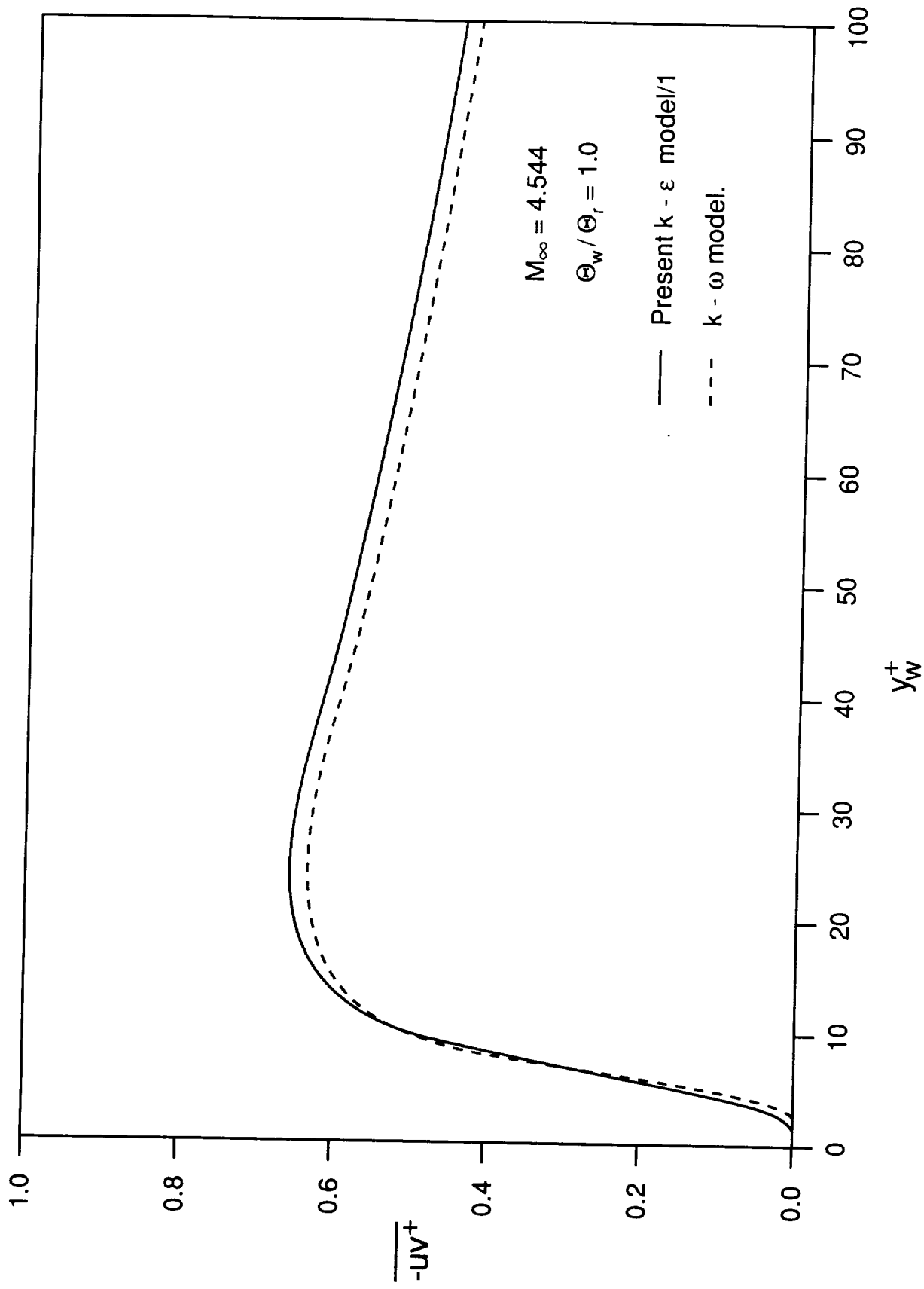


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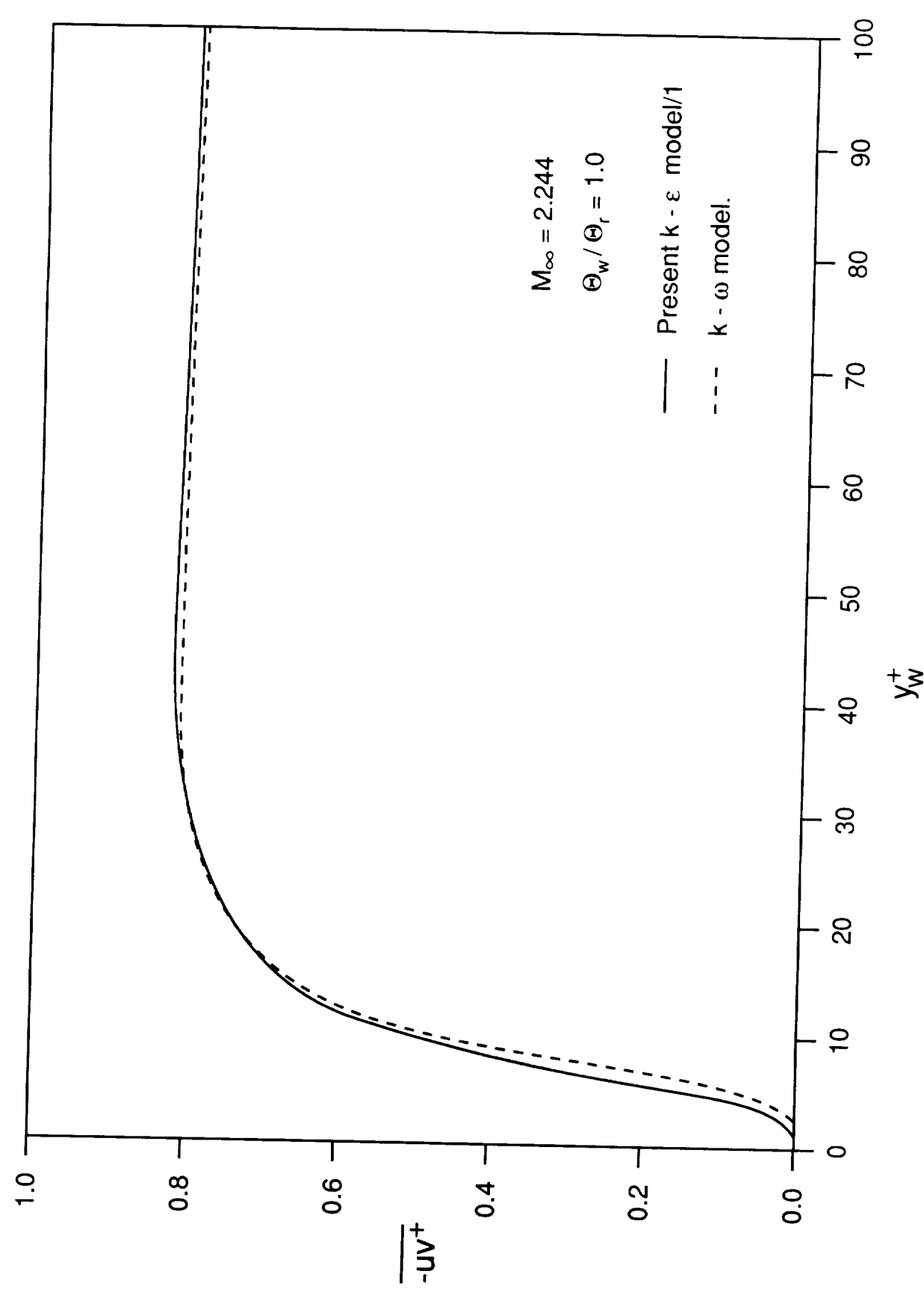


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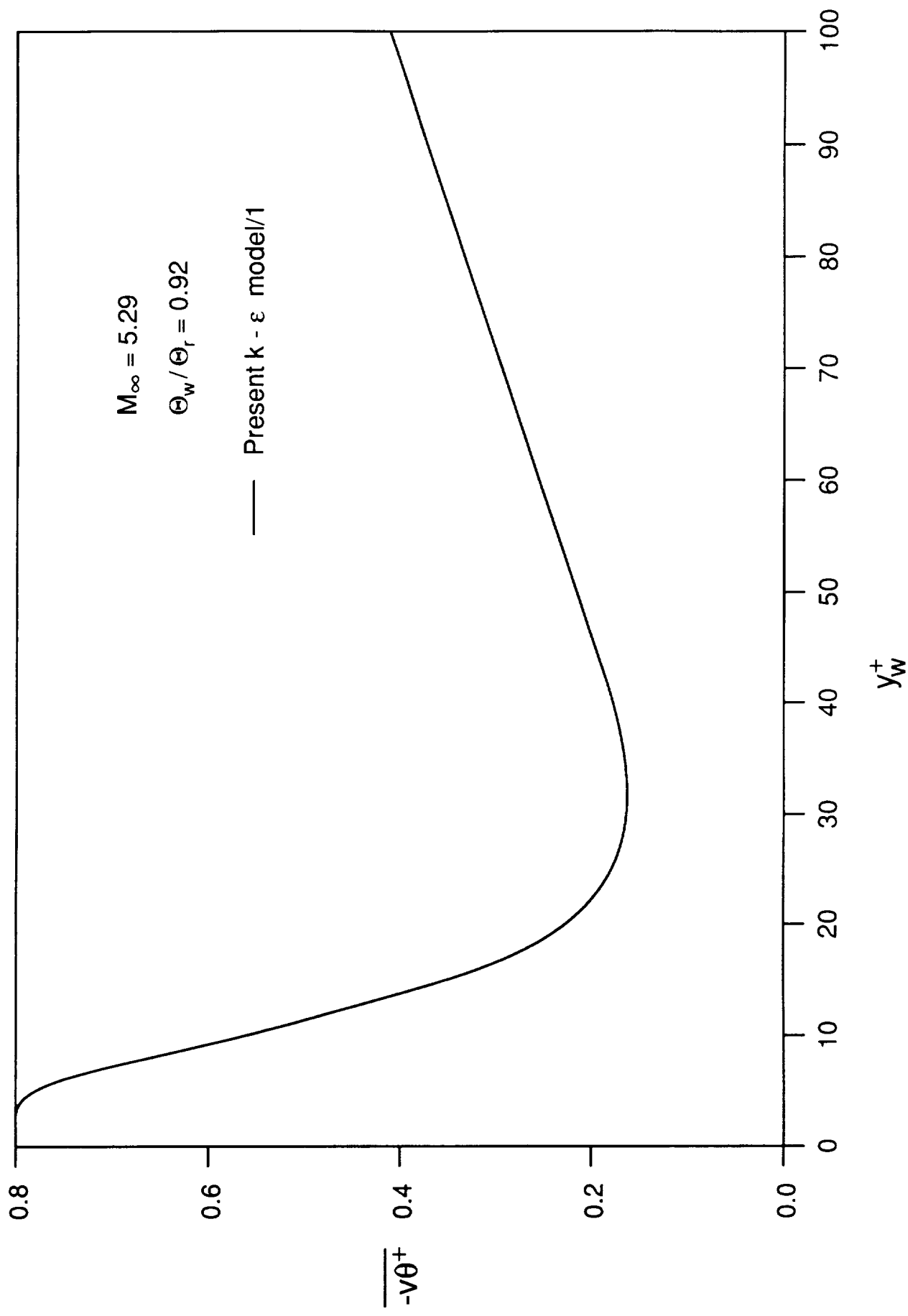


Figure 6a Near-wall distributions of Reynolds heat flux.



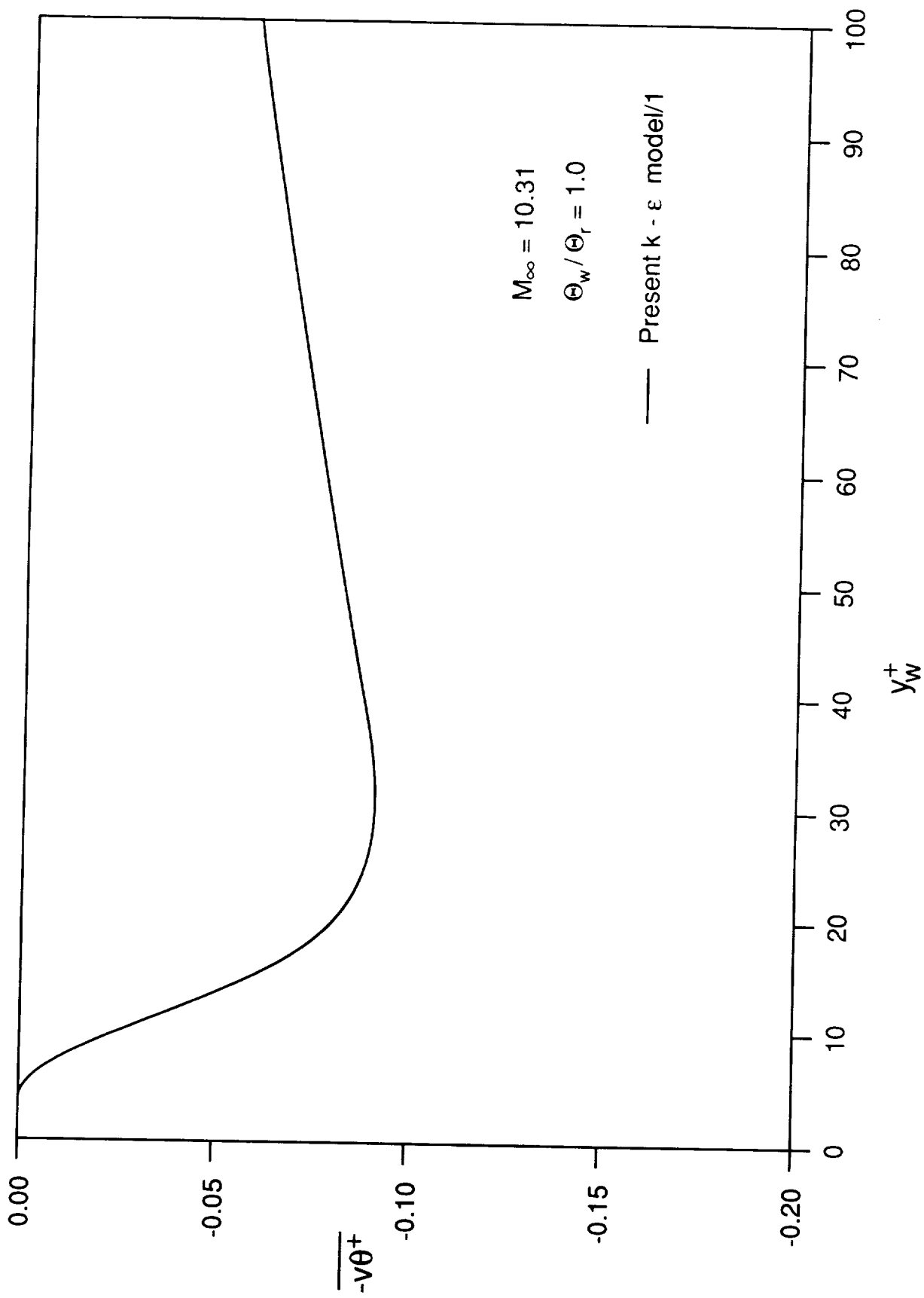


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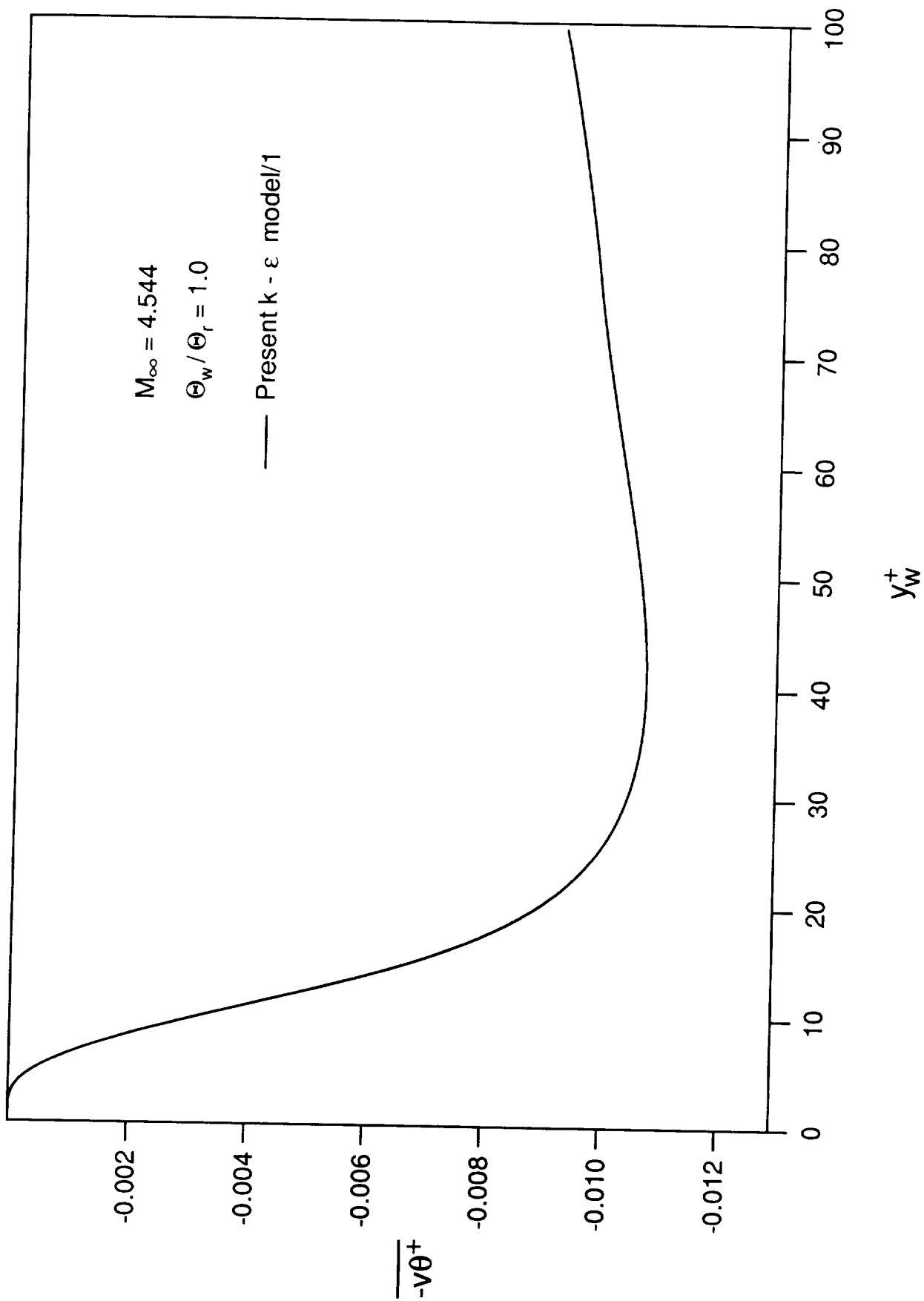


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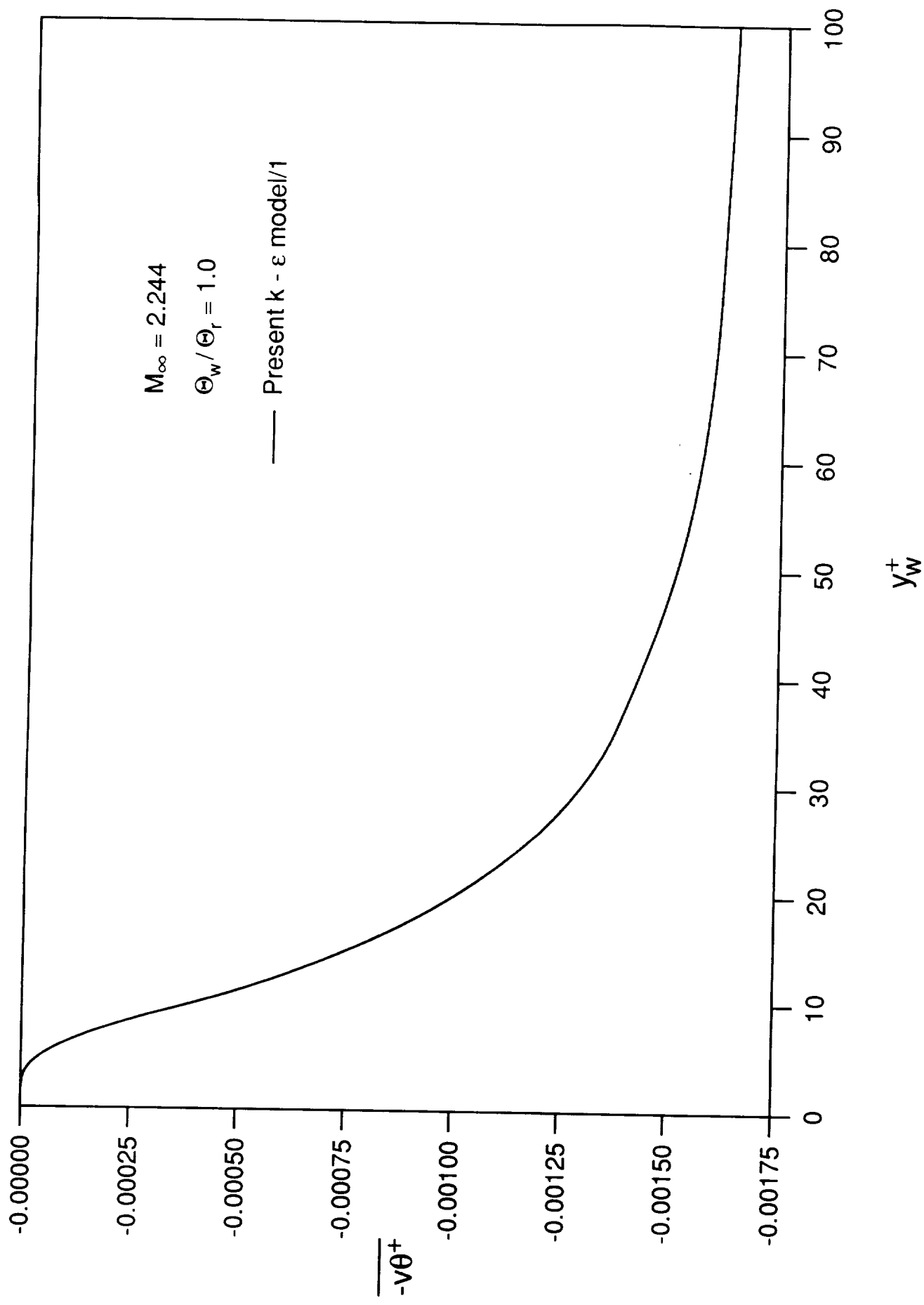


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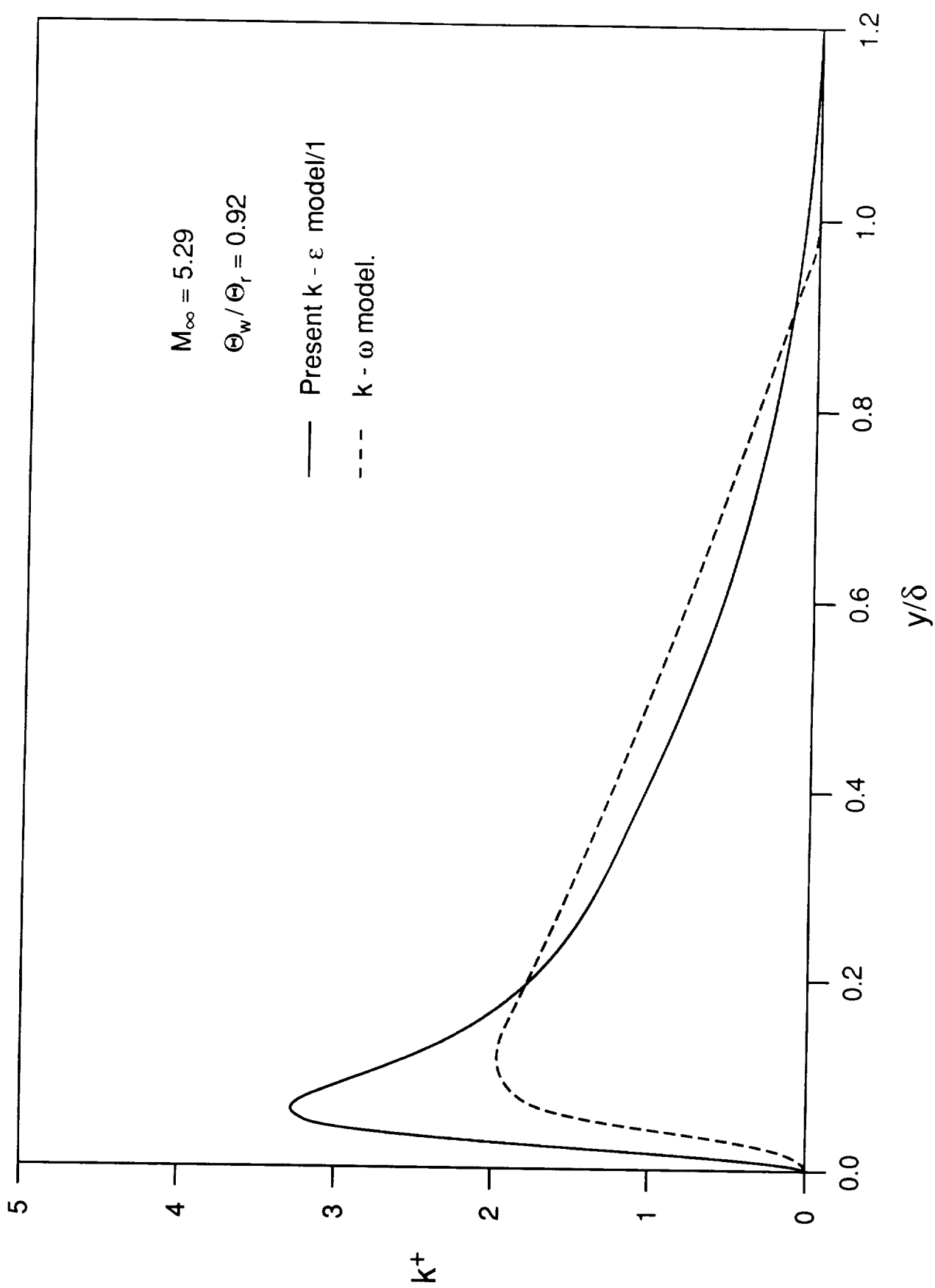


Figure 7a Turbulent kinetic energy distributions across the boundary layer.

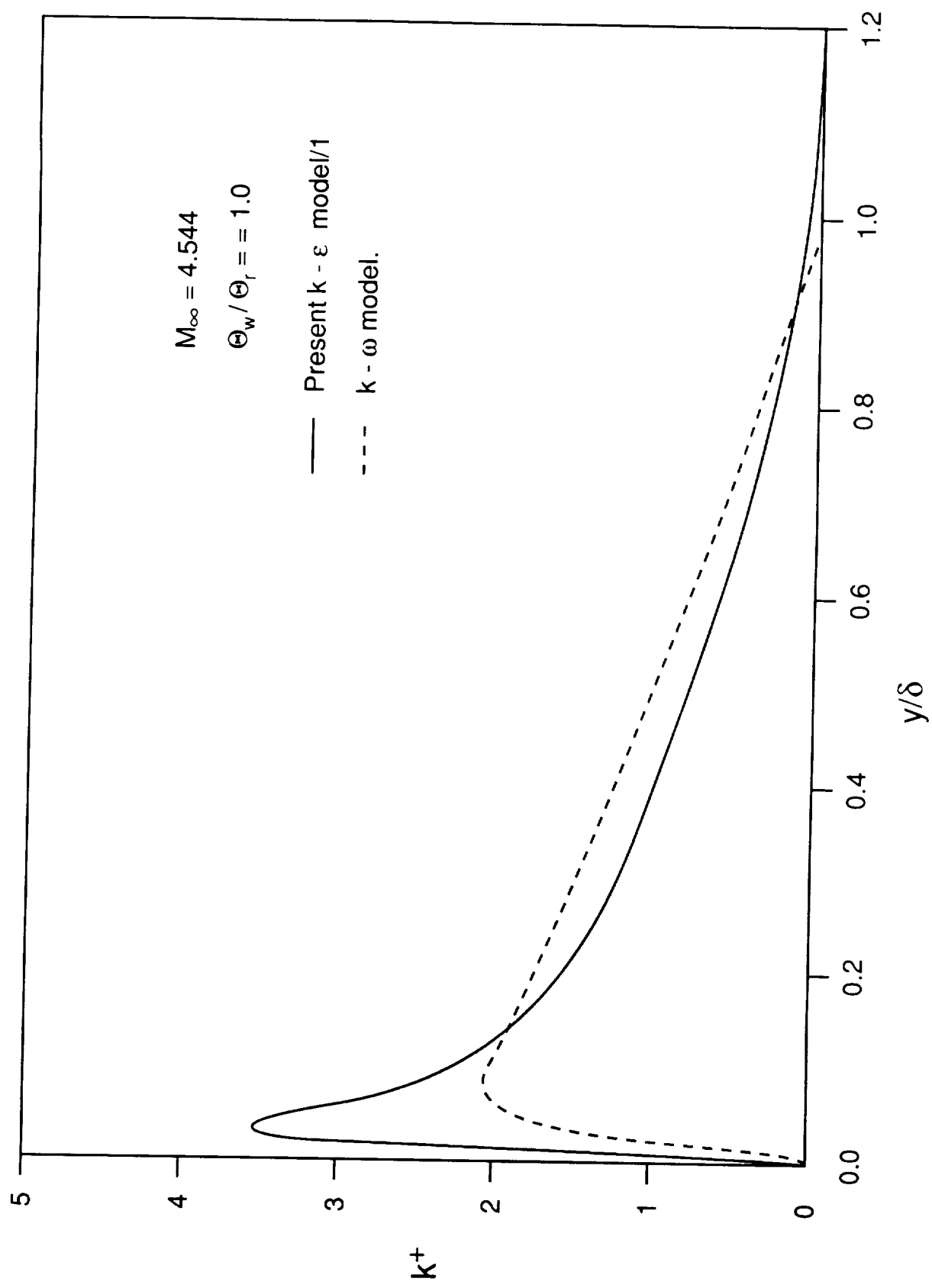


Figure 7b

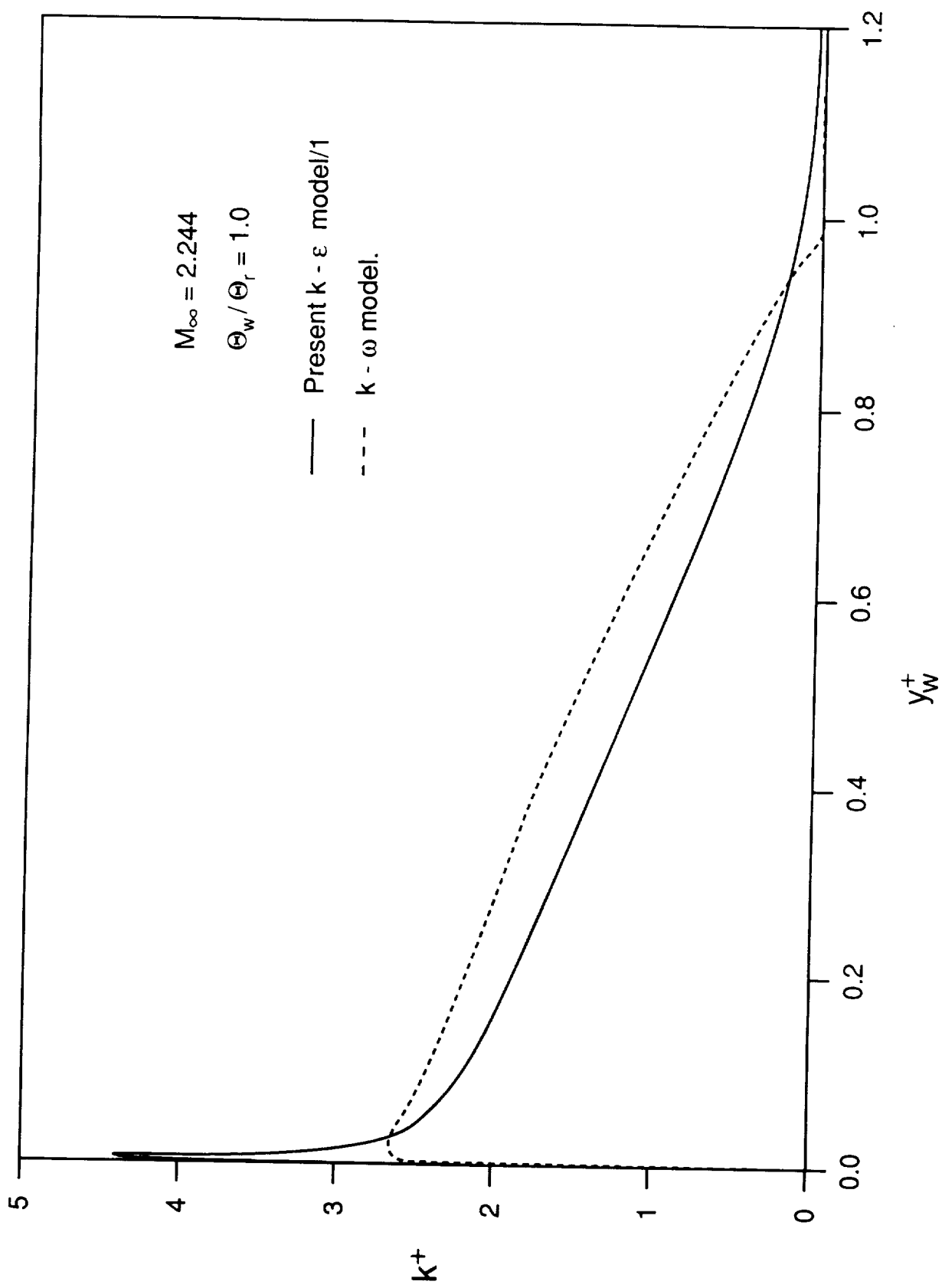


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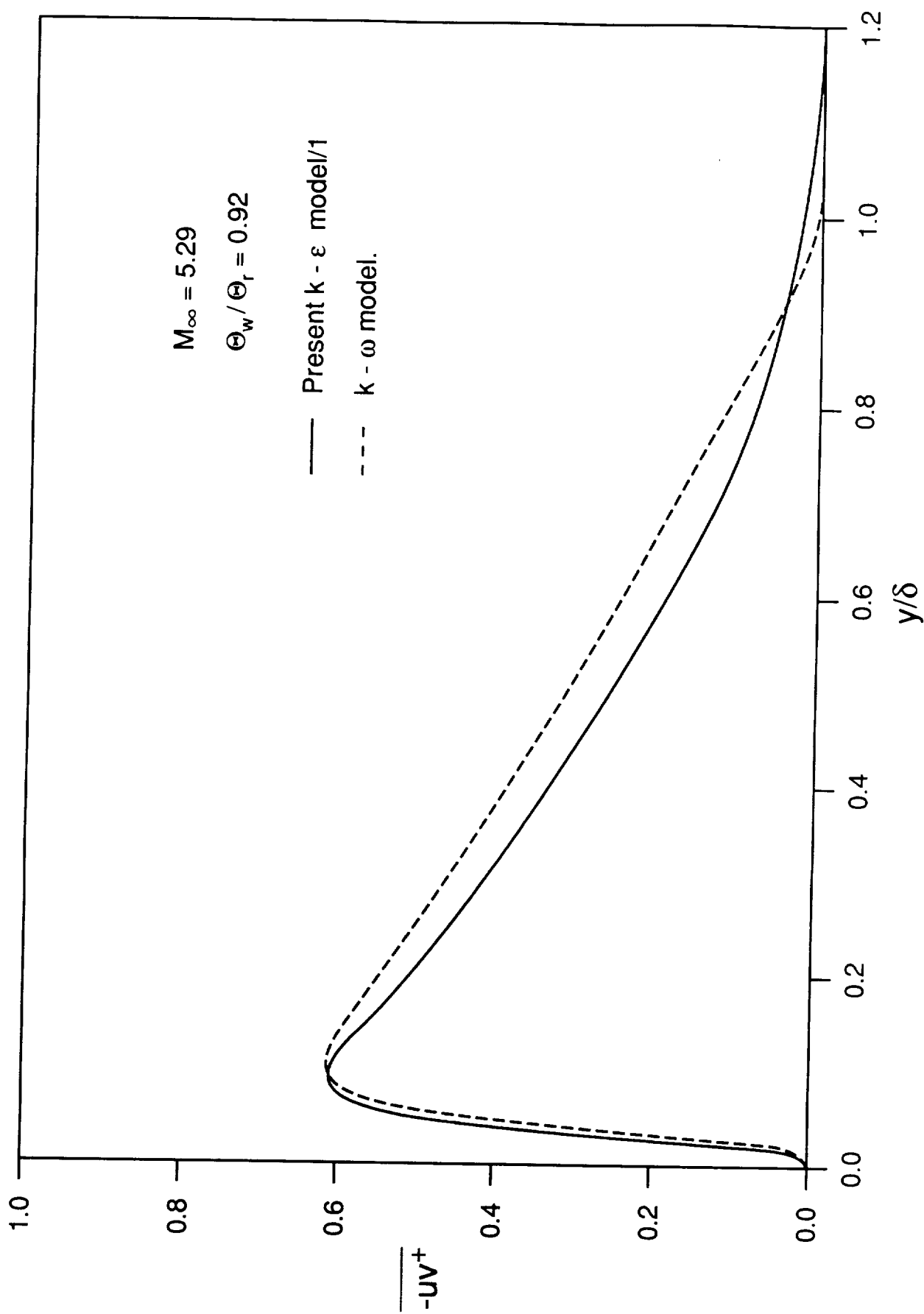


Figure 8a Reynolds shear stress distributions across the boundary layer.

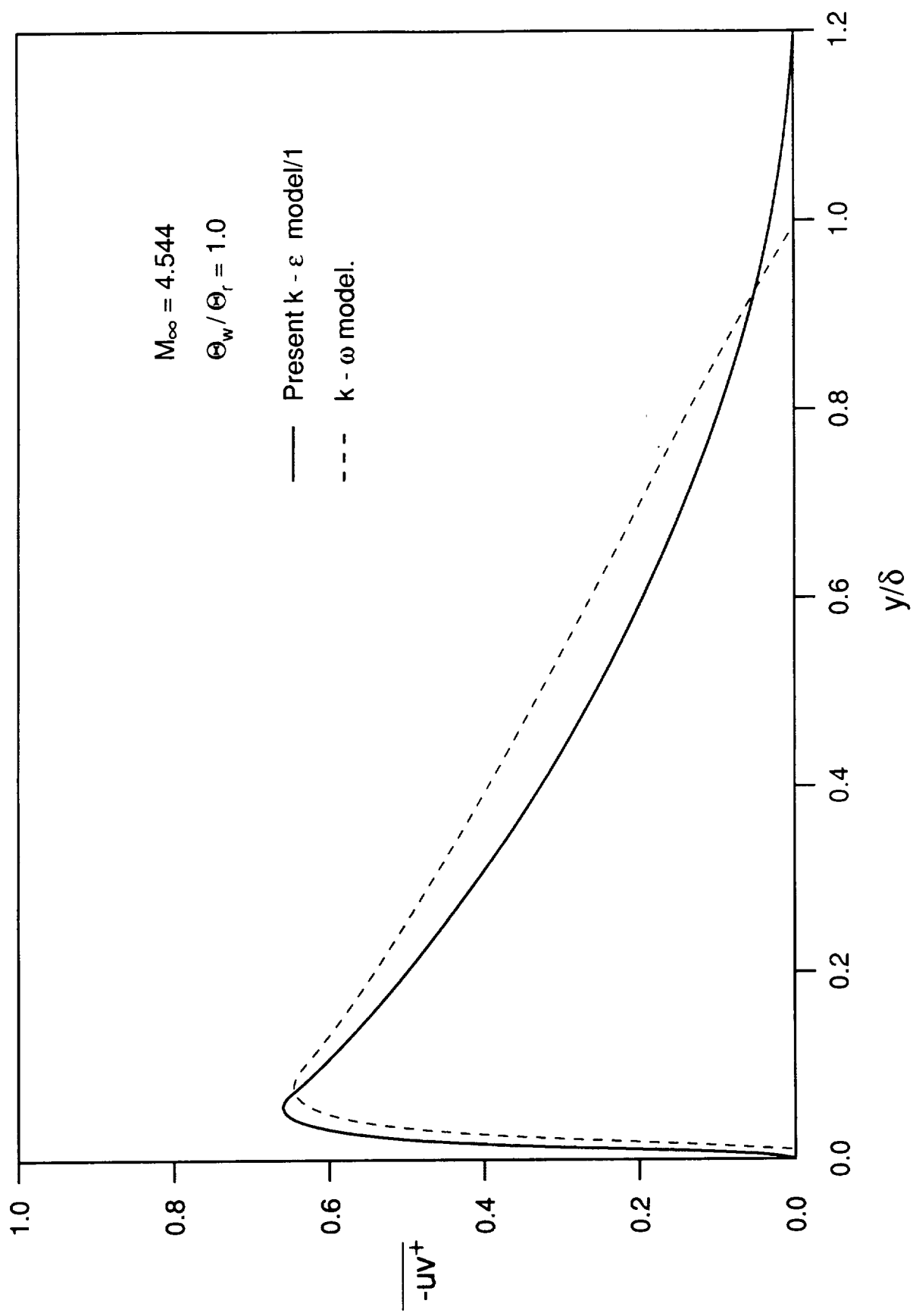


Figure 8b



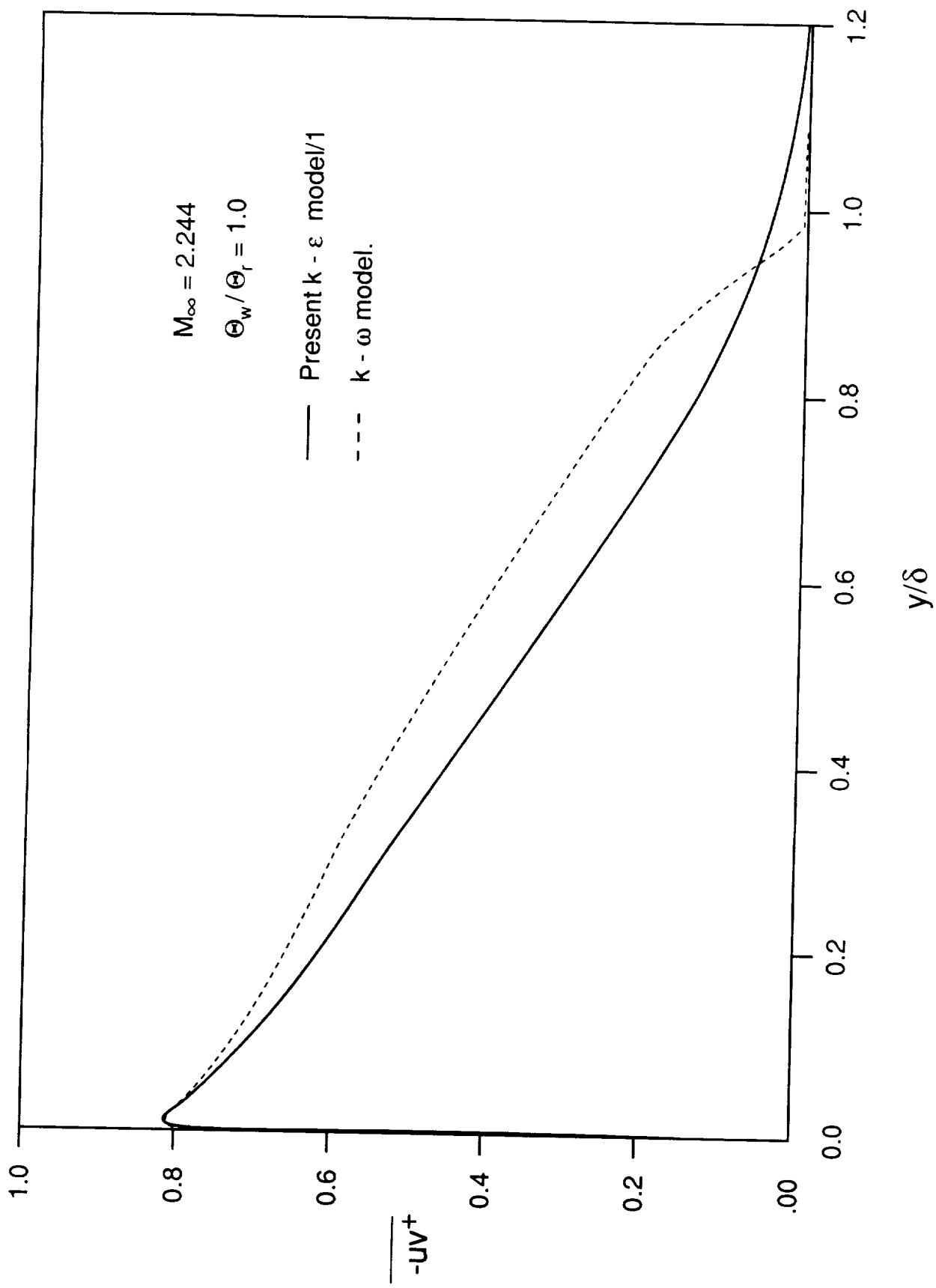


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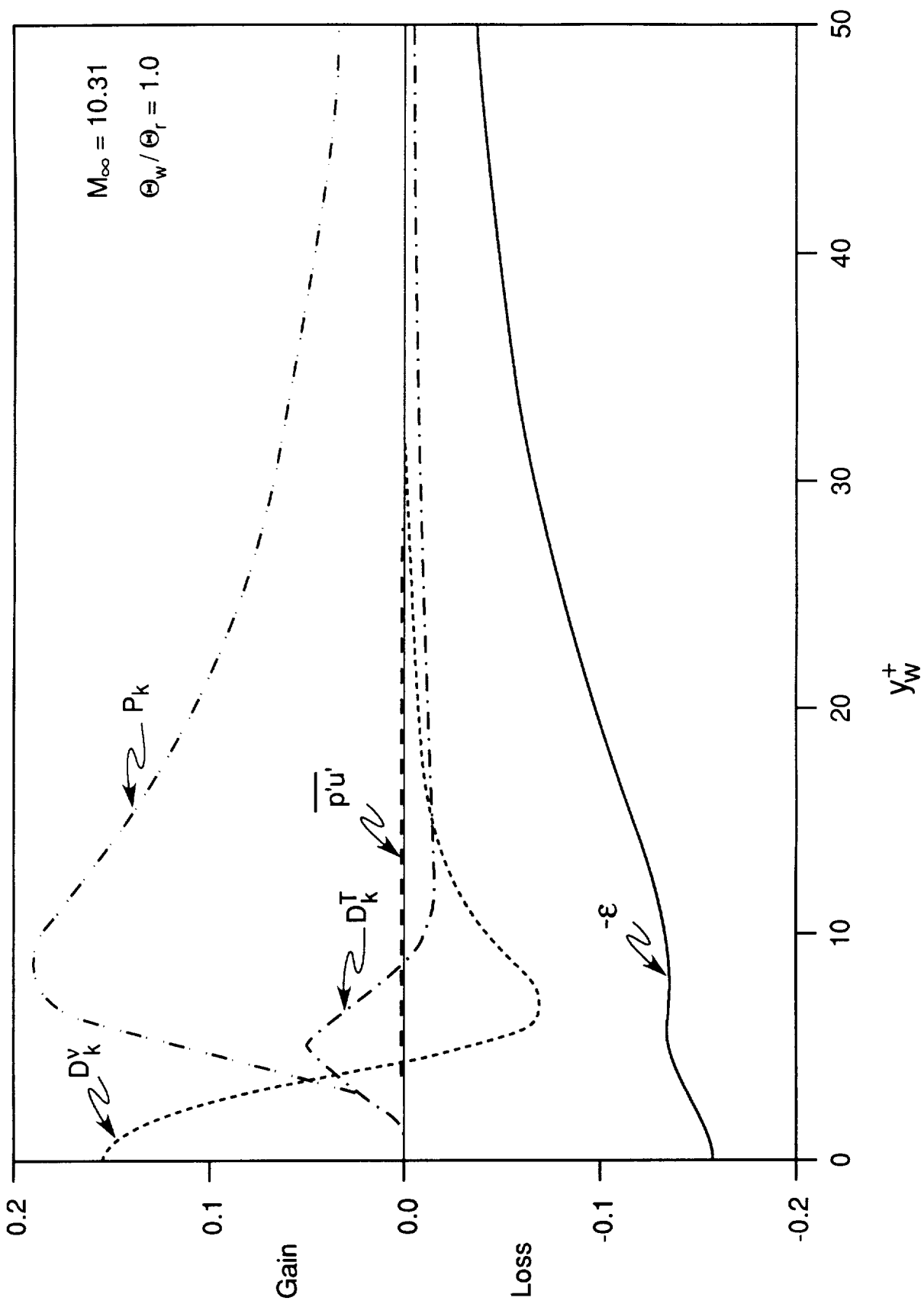


Figure 9 Near-wall  $k$  budget for the case with  $M_\infty = 10.31$  adiabatic wall boundary condition.

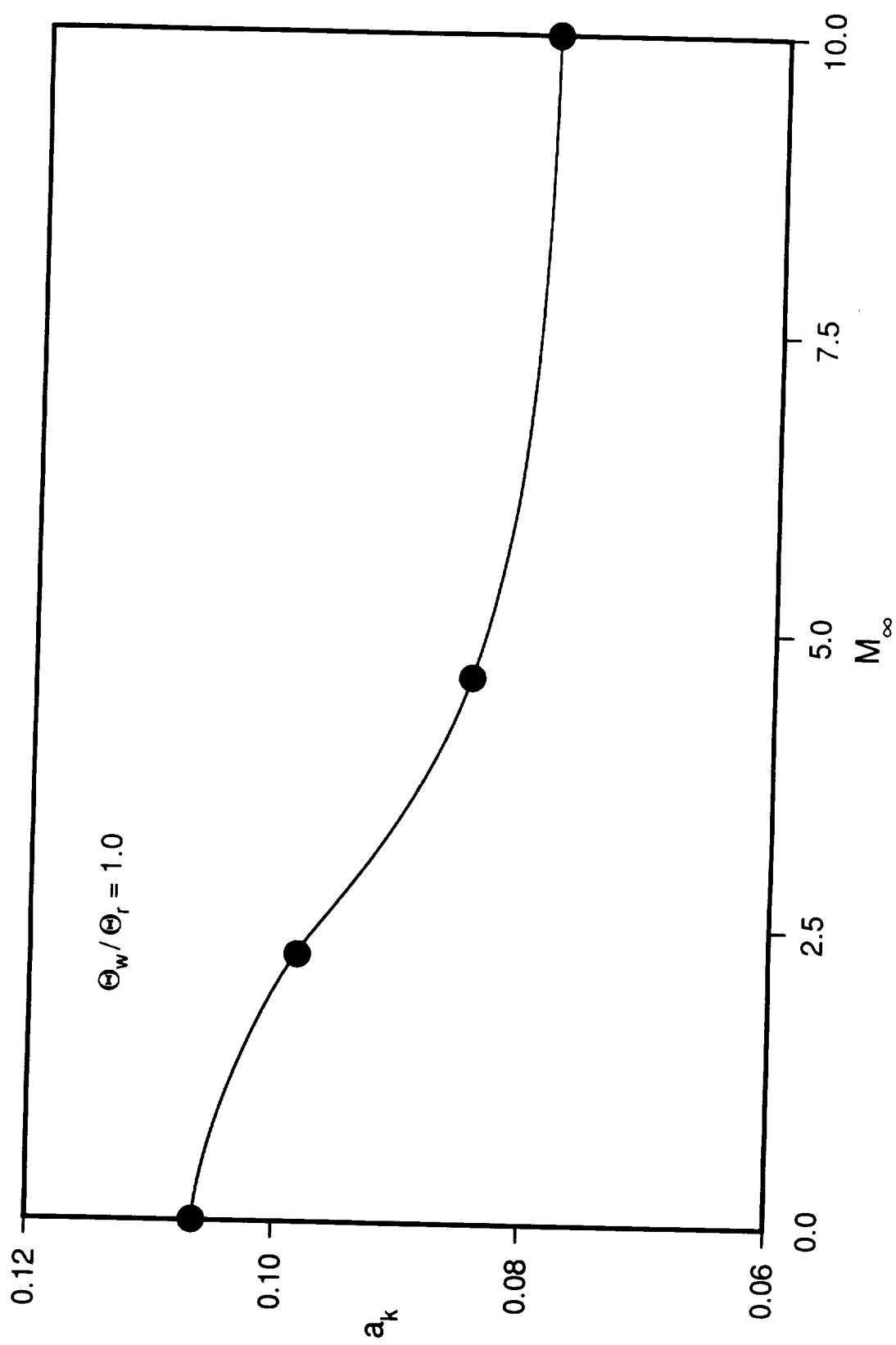


Figure 10 Variation of the slope of  $k^+$  at the wall with  $M_\infty$ .



**Figure 11** Effects of Reynolds number on model predictions for flows with adiabatic wall boundary condition.

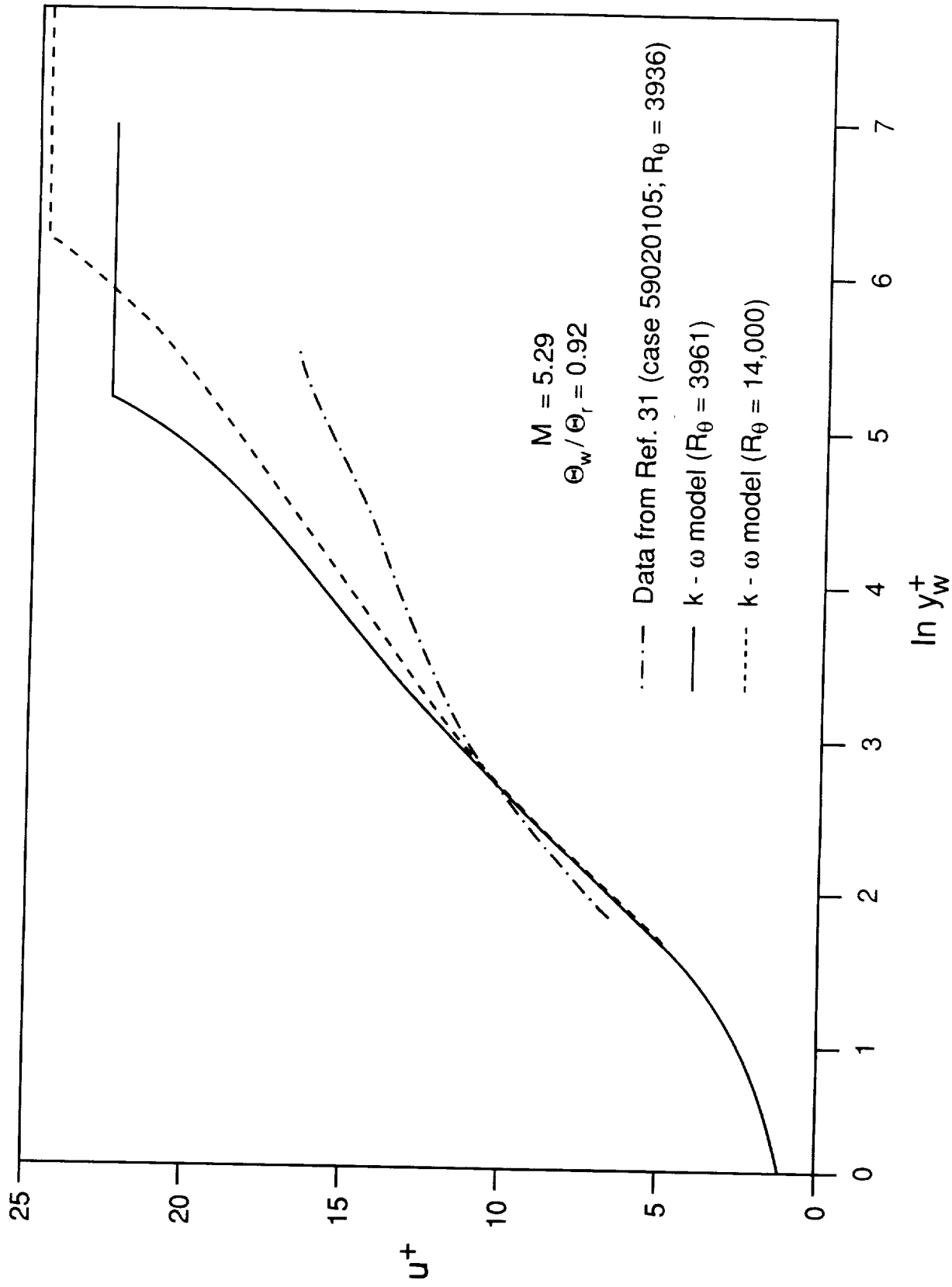


Figure 12a Effects of Reynolds number on model predictions for flows with constant wall temperature.

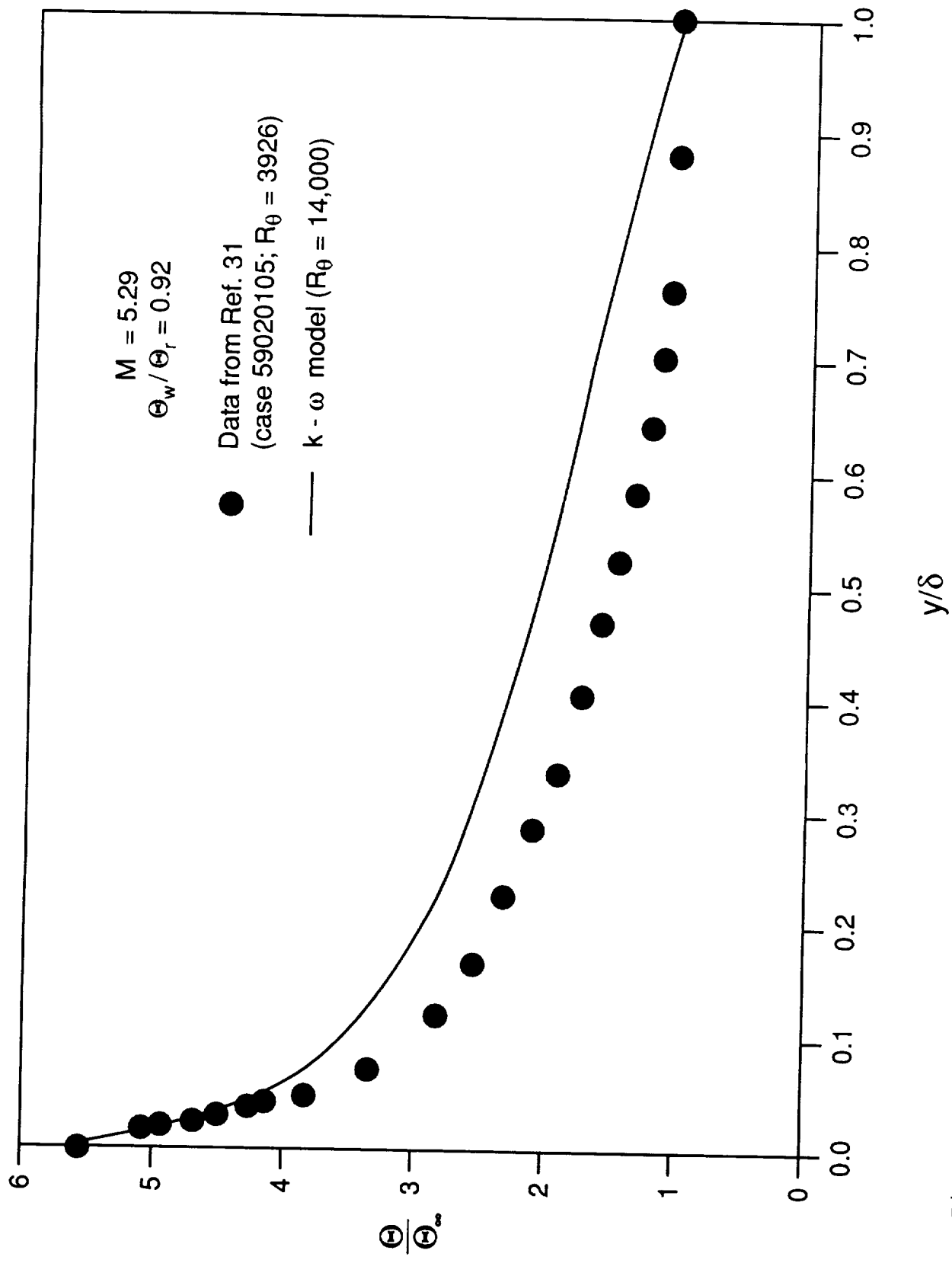


Figure 12b

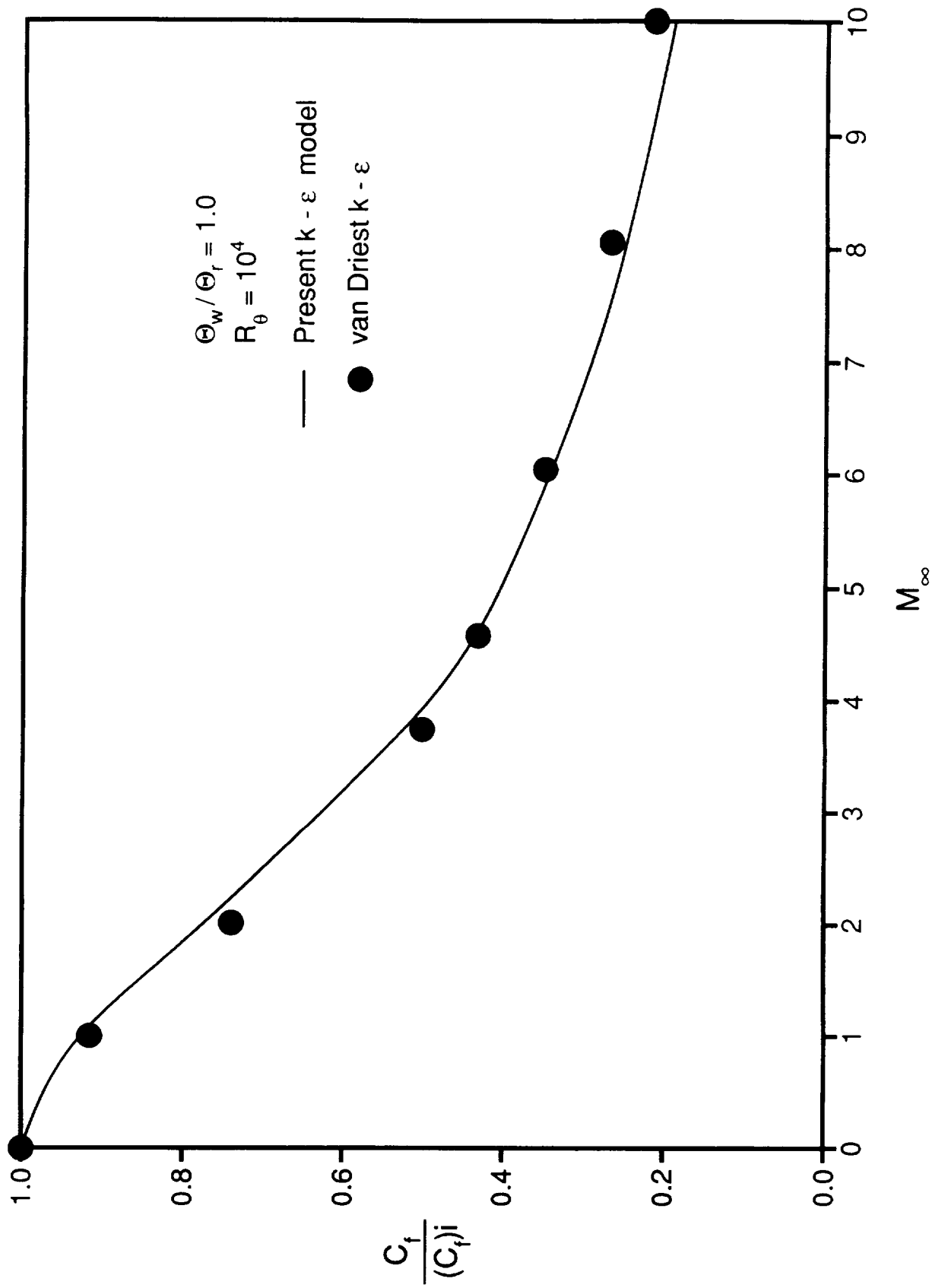


Figure 13 Skin friction coefficient variation with  $M_\infty$  for adiabatic wall boundary condition.